k-extendibility of high-dimensional bipartite quantum states

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Cécilia Lancien
Université Claude Bernard Lyon 1 & Universitat Autònoma de Barcelona

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1 The k-extendibility criterion for separability

Definition 1.1 (Separability vs entanglement). *A state ϱ_{AB} on $\mathrm{A}\otimes\mathrm{B}$ is separable if it may be written as a convex combination of product states, i.e. $\varrho_{AB}=\sum_x p_x\,\sigma_{\mathrm{A}}^x\otimes\tau_{\mathrm{B}}^x$. Otherwise, it is entangled.*

Deciding whether a given bipartite state is entangled or (close to) separable is an important issue in quantum physics. It is however, in general, a hard task (both from a mathematical and a computational point of view).
$\rightarrow$ Solution: Find set of states which are easier to characterize and which contain the set of separable states, hence providing NC for separability that may be checked efficiently (e.g. by an SDP).

Definition 1.2 (k-extendibility). *Let $k\in\mathbf{N}$. A state ϱ_{AB} on $\mathrm{A}\otimes\mathrm{B}$ is k-extendible w.r.t. B if there exists a state ϱ_{AB^k} on $\mathrm{A}\otimes\mathrm{B}^{\otimes k}$ which is invariant under any permutation of the B subsystems and s.t. $\varrho_{\mathrm{AB}}=\mathrm{Tr}_{\mathrm{B}^{k-1}}\varrho_{\mathrm{AB}^k}$.*

Theorem 1.3 (The complete family of k-extendibility criteria for separability, [7]). *On a bipartite Hilbert space $\mathrm{A}\otimes\mathrm{B}$, a state is separable if and only if it is k-extendible w.r.t. B for all $k\in\mathbf{N}$.*

Remarks: $\mathcal{S}_{\mathrm{A:B}}$ separable states. $\mathcal{E}^k_{\mathrm{A:B}}$ k-extendible states w.r.t. B.
• "$\varrho_{\mathrm{AB}}\in\mathcal{S}_{\mathrm{A:B}}\ \Rightarrow\ \forall\,k\in\mathbf{N},\ \varrho_{\mathrm{AB}}\in\mathcal{E}^k_{\mathrm{A:B}}$":
obvious since $\sigma_{\mathrm{A}}\otimes\tau_{\mathrm{B}}=\mathrm{Tr}_{\mathrm{B}^{k-1}}\left[\sigma_{\mathrm{A}}\otimes\tau_{\mathrm{B}}^{\otimes k}\right]$.
• "$\forall\,k\in\mathbf{N},\ \varrho_{\mathrm{AB}}\in\mathcal{E}^k_{\mathrm{A:B}}\ \Rightarrow\ \varrho_{\mathrm{AB}}\in\mathcal{S}_{\mathrm{A:B}}$":
relies on the quantum de Finetti theorem [6].
• $\varrho_{\mathrm{AB}}\in\mathcal{E}^k_{\mathrm{A:B}}\ \Rightarrow\ \forall\,k'\leqslant k,\ \varrho_{\mathrm{AB}}\in\mathcal{E}^{k'}_{\mathrm{A:B}}$.
$\rightarrow$ Sequence of increasingly constraining separability tests, which an entangled state is guaranteed to stop passing at some point.

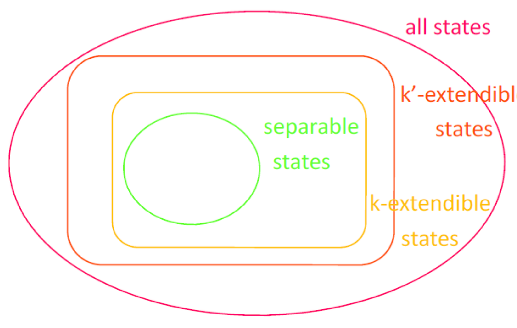

Figure 1: Nested separability relaxations

Problem: When relaxing separability to k-extendibility, how "rough" is the approximation?
Quantitative versions of the k-extendibility criterion: For any state ϱ_{AB} on $\mathrm{A}\otimes\mathrm{B}$,

$$\varrho_{\mathrm{AB}}\in\mathcal{E}^k_{\mathrm{A:B}}\ \Rightarrow\ \|\varrho_{\mathrm{AB}}-\mathcal{S}_{\mathrm{A:B}}\|_1\leqslant 2d_{\mathrm{B}}^2/k\ \ [6]\ \text{ and }\ \|\varrho_{\mathrm{AB}}-\mathcal{S}_{\mathrm{A:B}}\|_{\mathrm{LOCC}^{\rightarrow}}\leqslant\sqrt{2\ln d_{\mathrm{A}}/k}\ \ [5].$$

Bounds which are non-trivial only if $k\gg d_{\mathrm{B}}^2$ or $k\gg\ln d_{\mathrm{A}}$, so can anything interesting be said for a "not too big" k in the case where $d_{\mathrm{A}},d_{\mathrm{B}}$ are "big"? Nevertheless, they are known to be close from optimal: there exist states which are k-extendible, and nevertheless far away from the set of separable states in some standard or operational distance measure, hence "very" entangled.
$\rightarrow$ Instead of looking at worst case scenarios, can we make stronger statements about average/typical behaviours? Hope: Implications regarding average case complexity of checking separability...?

Two possible quantitative strategies:
• Estimate the size of the set of states, either satisfying a given separability criterion or being indeed separable.
$\rightarrow$ Information on how much bigger than the separable set the relaxed set is.
• Characterize when certain random states are with high probability, either violating a given separability criterion or indeed entangled. $\rightarrow$ Information on how powerful the separability test is to detect entanglement.

One useful observation: For any Hermitian M_{AB} on $\mathrm{A}\otimes\mathrm{B}$, $\sup_{\sigma_{\mathrm{AB}}\in\mathcal{E}^k}\mathrm{Tr}(M_{\mathrm{AB}}\sigma_{\mathrm{AB}})=\left\|\widetilde{M}_{\mathrm{AB}^k}\right\|_\infty$, where $\widetilde{M}_{\mathrm{AB}^k}=\frac{1}{k}\sum_{j=1}^k M_{\mathrm{AB}_j}\otimes\mathrm{Id}_{\mathrm{B}^k\setminus\mathrm{B}_j}$.

2 Technical interlude: GUE and Wishart matrices

Definition 2.1.
• *$n\times n$ GUE matrix: $G=(H+H^\dagger)/\sqrt{2}$ with H a $n\times n$ matrix having independent complex normal entries.*
• *(n,s)-Wishart matrix: $W=HH^\dagger$ with H a $n\times s$ matrix having independent complex normal entries.*

Definition 2.2.
• *Semicircular distribution of variance σ^2: $\mathrm{d}\mu_{SC(\sigma^2)}(x)=\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}\,\mathbf{1}_{[-2\sigma,2\sigma]}(x)\mathrm{d}x$.*
• *Marčenko-Pastur distribution of parameter λ: $\mathrm{d}\mu_{MP(\lambda)}(x)=\begin{cases}f_\lambda(x)\mathrm{d}x\ \text{if}\ \lambda>1\\(1-\lambda)\,\delta_0+\lambda f_\lambda(x)\mathrm{d}x\ \text{if}\ \lambda\leqslant 1\end{cases}$, where $f_\lambda(x)=\frac{\sqrt{(\lambda_+-x)(x-\lambda_-)}}{2\pi\lambda x}\mathbf{1}_{[\lambda_-,\lambda_+]}(x)$, with $\lambda_\pm=(\sqrt{\lambda}\pm 1)^2$.*

Link: For any Hermitian M on $\mathbf{C}^n$, denote by $N_M=\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(M)}$ its spectral distribution.
• $(G_n)_{n\in\mathbf{N}}$ sequence of $n\times n$ GUE matrices: $\left(N_{G_n/\sqrt{n}}\right)_{n\in\mathbf{N}}$ converges to $\mu_{SC(1)}$.
• $(W_n)_{n\in\mathbf{N}}$ sequence of $(n,\lambda n)$-Wishart matrices: $\left(N_{W_n/\lambda n}\right)_{n\in\mathbf{N}}$ converges to $\mu_{MP(\lambda)}$.

3 Mean width of the set of k-extendible states

Definition 3.1 (Mean-width of a set of states).
K convex set of states on $\mathbf{C}^n$, containing Id/n.
• *Width of K in the direction Δ, for Δ a $n\times n$ Hermitian having unit Hilbert-Schmidt norm: $w(K,\Delta):=\sup_{\sigma\in K}\mathrm{Tr}(\Delta(\sigma-\mathrm{Id}/n))$.*
• *Mean-width of K: $w(K):=\mathbf{E}\,w(K,\Delta)$, for Δ uniformly distributed on the Hilbert-Schmidt norm unit sphere.*
Equivalently: $w(K)\sim_{n\to+\infty}\mathbf{E}\,w(K,G)/n$, for G a $n\times n$ GUE matrix.

$\rightarrow$ The mean-width of K is a certain measure of its size (for any "reasonable" K, $w(K)\simeq vrad(K)$, where $vrad(K)$ is the volume-radius of K, i.e. the radius of the Euclidean ball with same volume as K).

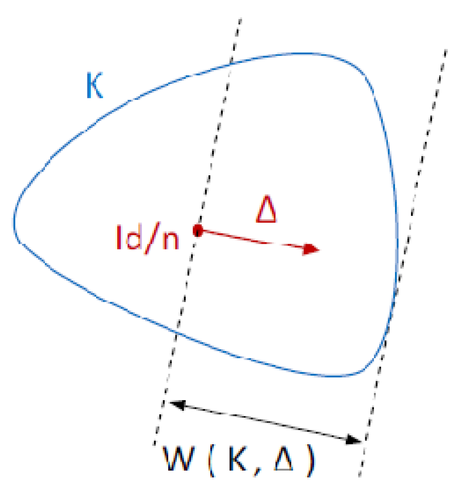

Figure 2: Width of K in the direction Δ

Theorem 3.2. *On $\mathbf{C}^d\otimes\mathbf{C}^d$, denote by $\mathcal{S}$, resp. $\mathcal{E}^k$, the set of separable, resp. k-extendible, states. Then,*

$$\frac{c}{d^{3/2}}\leqslant w(\mathcal{S})\leqslant\frac{C}{d^{3/2}},\ c,C>0\ \textit{universal constants (c.f. [3]), while}\ w(\mathcal{E}^k)\underset{d\to+\infty}{\sim}\frac{2}{\sqrt{k}d}.$$

For $k\in\mathbf{N}$ fixed, the mean-width of the set of k-extendible states is of order $1/d$, hence much bigger than the one of the set of separable states.
$\rightarrow$ On high dimensional bipartite systems, the set of k-extendible states is a very rough approximation of the set of separable states.

Main steps in the proof:
$\mathbf{E}\sup_{\sigma\in\mathcal{E}^k}\mathrm{Tr}(G(\sigma-\mathrm{Id}/d^2))=\mathbf{E}\|\widetilde{G}\|_\infty$. To estimate the operator norm of the "modified" GUE matrix $\widetilde{G}$: compute all p-order moments $\mathbf{E}\,\mathrm{Tr}\,\widetilde{G}^p$, and identify the limiting spectral distribution. After rescaling by d/k: a semicircular distribution $\mu_{SC(k)}$. The latter's support has $2\sqrt{k}$ as upper-edge.

4 k-extendibility of random induced quantum states

Definition 4.1 (Random induced states). *System space $\mathrm{H}\equiv\mathbf{C}^n$. Ancilla space $\mathrm{H}'\equiv\mathbf{C}^s$.*
$\rightarrow$ Random mixed state model on H: $\varrho=\mathrm{Tr}_{\mathrm{H}'}|\psi\rangle\langle\psi|$ with $|\psi\rangle$ a uniformly distributed pure state on $\mathrm{H}\otimes\mathrm{H}'$.
Equivalently: $\varrho=W/\mathrm{Tr}\,W$, with W a (n,s)-Wishart matrix.

Question: Fix $d\in\mathbf{N}$ and consider ϱ a random state on $\mathbf{C}^d\otimes\mathbf{C}^d$ induced by some environment $\mathbf{C}^s$.
For which values of s is ϱ typically separable/k-extendible?
"typically" = "with probability going to 1 as d grows". Hence, two steps:
(a) Identify the range of s where ϱ is, on average, separable/k-extendible.
(b) Show that the average behaviour is generic in high dimension (concentration of measure phenomenon: a sufficiently "well-behaved" function has an exponentially small probability of deviating from its average as the dimension grows).

Theorem 4.2. *Let ϱ be a random state on $\mathbf{C}^d\otimes\mathbf{C}^d$ induced by some environment $\mathbf{C}^s$.*
• *$s<cd^3\Rightarrow\varrho$ typically entangled, and $s>Cd^3\log^2 d\Rightarrow\varrho$ typically separable (c.f. [4]).*
• *$s<\frac{(k-1)^2}{4k}d^2\Rightarrow\varrho$ typically not k-extendible, and $s>C_kd^2\log^2 d\Rightarrow\varrho$ typically k-extendible.*
$c,C>0$ universal constants, $C_k>0$ constant which depends on k.

For $k\in\mathbf{N}$ fixed, the threshold environment dimension at which random induced states are generically k-extendible or not is of order d^2 (up to $\log$ factors), hence much smaller than the one at which they are generically separable or not.
$\rightarrow$ In the range $d^2\lesssim s\lesssim d^3$, the typical entanglement of random induced states is typically not detected by the k-extendibility test.

Main steps in the proof:
• **Lower-bound:** If $\sup_{\sigma\in\mathcal{E}^k}\mathrm{Tr}(\varrho\sigma)<\mathrm{Tr}(\varrho^2)$, then $\varrho\notin\mathcal{E}^k$ (non-k-extendibility witness). Hence, characterize when $\mathbf{E}\sup_{\sigma\in\mathcal{E}^k}\mathrm{Tr}(W\sigma)<\mathbf{E}\,\mathrm{Tr}(W^2)/\mathbf{E}\,\mathrm{Tr}\,W$ for W a (d^2,s)-Wishart matrix.
RHS: In the limit $d,s\to+\infty$, $\mathbf{E}\,\mathrm{Tr}(W^2)=d^4s+d^2s^2$ and $\mathbf{E}\,\mathrm{Tr}\,W=d^2s$.
LHS: $\mathbf{E}\sup_{\sigma\in\mathcal{E}^k}\mathrm{Tr}(W\sigma)=\mathbf{E}\|\widetilde{W}\|_\infty$. To estimate the operator norm of the "modified" Wishart matrix $\widetilde{W}$: compute all p-order moments $\mathbf{E}\,\mathrm{Tr}\,\widetilde{W}^p$, and identify the limiting spectral distribution. After rescaling by s/k: a Marčenko-Pastur distribution $\mu_{MP(ks/d^2)}$. The latter's support has $(\sqrt{ks/d^2}+1)^2$ as upper-edge.
• **Upper-bound:** Relating the gauge of a sufficiently "well-balanced" convex body to that of its polar + Comparing averages of unitarily invariant norms over different random matrix ensembles (majorization).

5 Generalizations

• **Adding the constraint that the symmetric extension is PPT across one (even) cut:**
Other complete hierarchy of separability criteria, where the SDP to be solved is bigger but much faster [9]. However, imposing this extra requirement only reduces the mean-width of the set of k-extendible states by a factor $\sqrt{2}$...

• **Generalization to the unbalanced case:** $\mathrm{A}\equiv\mathbf{C}^{d_{\mathrm{A}}}$, $\mathrm{B}\equiv\mathbf{C}^{d_{\mathrm{B}}}$, $d_{\mathrm{A}}\neq d_{\mathrm{B}}$
Straightforward if $d_{\mathrm{A}},d_{\mathrm{B}}\to+\infty$, but more subtle if d_{A} or d_{B} is fixed (free probability approach perhaps more relevant and powerful in that setting...?)

• **What happens if k is not fixed, but instead grows with** d? Partial answers only...
If $1\ll k\ll d$, then $w(\mathcal{S})\ll w(\mathcal{E}^k)\ll w(\mathcal{D})$ (i.e. $\mathcal{E}^k$ lies "strictly in between" $\mathcal{S}$ and $\mathcal{D}$).
$\rightarrow k\gtrsim d$ is necessary to have $w(\mathcal{E}^k)\simeq w(\mathcal{S})$.

6 Summary and perspectives

When $k\geqslant 2$ is a fixed parameter, asymptotic weakness of the k-extendibility NC for separability (as the dimensions of the underlying local Hilbert spaces grow).
Similar features are exhibited by all other known separability criteria, such as PPT [1], realignment [2], reduction [8] etc.

from the point of view of / k-extendibility "beats"	average size	entanglement detection of random states
PPT	for $k\geqslant 11$	for $k\geqslant 17$
realignment	?	for $k\geqslant 5$

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