Typical correlations and entanglement in random tensor network states

Based mostly on joint works with:
• David Pérez-García (*Ann. Henri Poincaré* 23(1):141-222, 2022)
• Léo Le Nestour and David Pérez-García (in progress)

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`Institut Fourier Grenoble & CNRS`

QMP Intensive Period (QIEC Workshop) – May 9 2025

Outline

1. Background and motivations
2. Random MPS and PEPS: construction and statement of the main questions & results
3. Typical correlation length in a random TNS through typical spectral gap of its transfer operator
4. Typical correlation length in a random TNS through typical spectral gap of its parent Hamiltonian
5. Open questions and perspectives

The curse of dimensionality when dealing with many-body quantum systems

A quantum system composed of N d-dimensional subsystems has dimension d^N.
$\longrightarrow$ Exponential growth of the dimension with the number of subsystems.
However, *'physically relevant' states* of many-body quantum systems are often well approximated by so-called *tensor network states (TNS)*, which form a small subset of the global state space.

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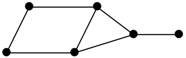
Tensor network state construction on $(\mathbf{C}^d)^{\otimes N}$:
Take a graph G with N vertices and P edges.
Put at each vertex v a tensor $|\Phi_v\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes e(v)}$ to get a tensor $|\Phi_\mathrm{G}\rangle \in (\mathbf{C}^d)^{\otimes N} \otimes (\mathbf{C}^q)^{\otimes 2P}$.
Contract together the indices of $|\Phi_\mathrm{G}\rangle$ associated to a same edge to get a tensor $|\varphi_\mathrm{G}\rangle \in (\mathbf{C}^d)^{\otimes N}$.

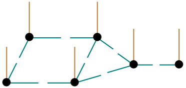
$e(v)$: number of edges at v

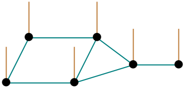
$|\Phi_\mathrm{G}\rangle = \bigotimes_v |\Phi_v\rangle$

$|\varphi_\mathrm{G}\rangle$: pure state on $(\mathbf{C}^d)^{\otimes N}$ (up to normalization)

$\longrightarrow$ If $e(v) \leqslant r$ for all v, such state is described by at most $Nq^r d$ parameters (linear in N).


G with 6 vertices and 7 edges


$|\Phi_\mathrm{G}\rangle \in (\mathbf{C}^d)^{\otimes 6} \otimes (\mathbf{C}^q)^{\otimes 14}$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d-dimensional indices: *physical* indices. q-dimensional indices: *bond* indices.

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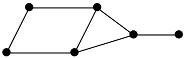

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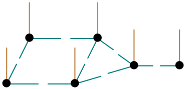

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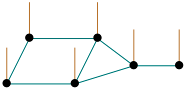

$|\varphi_{\mathrm{G}}\rangle \in (\mathbf{C}^d)^{\otimes 6}$

d-dimensional indices: *physical* indices. q-dimensional indices: *bond* indices.

In this talk: The underlying graph G is a regular lattice in dimension 1 or 2.
$\longrightarrow$ $|\varphi_{\mathrm{G}}\rangle$ is a *matrix product state (MPS)* or a *projected entangled pair state (PEPS)*.

Motivations behind TNS and related questions

Intuition: A TNS is meant to be a *mathematically tractable* description of the state of a system composed of *many subsystems* which are subject to interactions respecting a certain *geometry*.

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Example: Being a TNS is known/conjectured to be approximately equivalent to being a ground state of a gapped local Hamiltonian (Hastings, Landau/Vazirani/Vidick...)

↳ composed of terms which act non-trivially only on nearby sites

↳ spectral gap lower bounded by a constant independent of N

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General question: Among the various *qualitative* intuitions about TNS, which ones are at least true generically? And can *quantitative* statements be made about the generic case?

Idea: Sample a TNS at random and study the characteristics that it exhibits with high probability.
In particular, what is its *typical amount of correlations and entanglement*?
Regime we are interested in: fixed (large) d and q, ideally any N.

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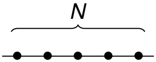
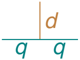
A simple model of random translation-invariant MPS

N subsystems on circle

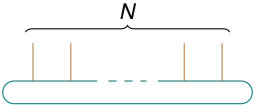


Pick a tensor $|\Phi\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes 2}$ whose entries are independent complex Gaussians with mean 0 and variance $1/dq$.
Repeat it on all sites and contract neighboring q-dimensional indices.
$\longrightarrow$ Obtained tensor $|\varphi_N\rangle \in (\mathbf{C}^d)^{\otimes N}$: *random translation-invariant MPS with periodic boundary conditions* (typically almost normalized).

1-site tensor



$$|\Phi\rangle = \sum_{x=1}^{d} \sum_{i,j=1}^{q} g_{xij} |xij\rangle$$



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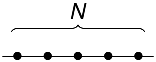

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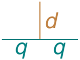

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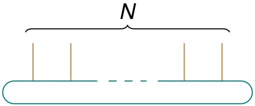

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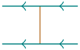
Associated *transfer operator*: $T : \mathbf{C}^q \otimes \mathbf{C}^q \to \mathbf{C}^q \otimes \mathbf{C}^q$, obtained by contracting the d-dimensional indices of $|\Phi\rangle$ and $|\bar{\Phi}\rangle$.

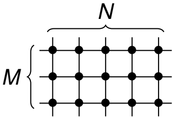


$$T = \sum_{x=1}^{d} \left(\sum_{i,j,k,l=1}^{q} g_{xij} \bar{g}_{xkl} |ik\rangle\langle jl| \right) = \frac{1}{d} \sum_{x=1}^{d} G_x \otimes \bar{G}_x$$

The G_x's are independent $q \times q$ matrices whose entries are independent complex Gaussians with mean 0 and variance $1/q$.

A simple model of random translation-invariant PEPS

MN subsystems on torus



Pick a tensor $|\Phi\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes 4}$ whose entries are independent complex Gaussians with mean 0 and variance $1/dq^2$.
Repeat it on all sites and contract neighboring q-dimensional indices.
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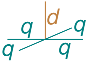
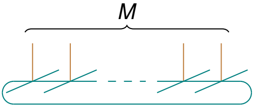
1-site tensor



$$|\Phi\rangle = \sum_{x=1}^{d} \sum_{i,j,i',j'=1}^{q} g_{xiji'j'} |xiji'j'\rangle$$

1-column tensor



$$|\Phi_M\rangle \in (\mathbf{C}^d \otimes \mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$$

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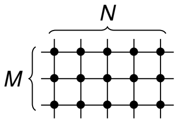

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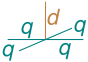

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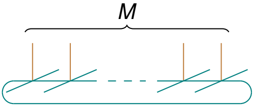

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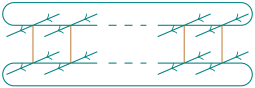
Associated *transfer operator*: $T_M : (\mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M} \to (\mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$, obtained by contracting the d-dimensional indices of $|\Phi_M\rangle$ and $|\bar{\Phi}_M\rangle$.



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The G_{xij}'s are independent $q \times q$ matrices whose entries are independent complex Gaussians with mean 0 and variance $1/q$.

Correlations in a TNS

$|\varphi\rangle \in (\mathbf{C}^d)^{\otimes MN}$ an MPS ($M = 1$) or PEPS ($M > 1$).
A, B local observables, i.e. on 1 subsystem $\mathbf{C}^d$ (or few neighboring subsystems).

Goal: Quantify the amount of correlations between the values of A and B, when measured on distant sites k, l of the TNS $|\varphi\rangle$.

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Compute the value on $|\varphi\rangle$ of the observable $A_k \otimes B_l \otimes \mathrm{I}_{[MN]\setminus\{k,l\}}$, i.e.

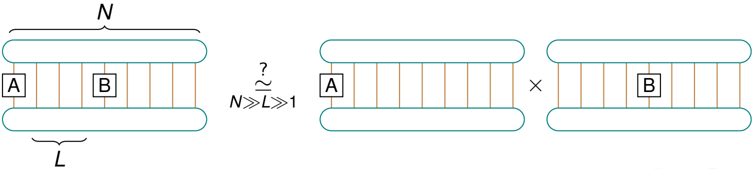
$$v_\varphi(A, B, k, l) := \langle\varphi|A_k \otimes B_l \otimes \mathrm{I}_{[MN]\setminus\{k,l\}}|\varphi\rangle.$$

Compare it to the product of values on $|\varphi\rangle$ of observables $A_k \otimes \mathrm{I}_{[MN]\setminus\{k\}}$ and $B_l \otimes \mathrm{I}_{[MN]\setminus\{l\}}$, i.e.

$$v_\varphi(A, k)v_\varphi(B, l) := \langle\varphi|A_k \otimes \mathrm{I}_{[MN]\setminus\{k\}}|\varphi\rangle\langle\varphi|B_l \otimes \mathrm{I}_{[MN]\setminus\{l\}}|\varphi\rangle.$$

Correlations in the TNS $|\varphi\rangle$: $\gamma_\varphi(A, B, k, l) := \big|v_\varphi(A, B, k, l) - v_\varphi(A, k)v_\varphi(B, l)\big|$.

Question: Do we have $\gamma_\varphi(A, B, k, l) \to 0$ as $\mathrm{dist}(k, l) \to \infty$? And if so, at which speed?



Main result: random TNS typically exhibit fast exponential decay of correlations

Intuition: For basically any TNS $|\varphi\rangle \in (\mathbf{C}^d)^{\otimes MN}$, the correlations between two 1-site observables decay exponentially with the distance separating the two sites, i.e. there exist $C(\varphi) < \infty$, $\tau(\varphi) > 0$ such that, for any sites k, l at distance $L \ll N$ and any observables A, B on $\mathbf{C}^d$,

$$\gamma_\varphi(A, B, k, l) \leqslant C(\varphi) e^{-\tau(\varphi) L} \|A\|_\infty \|B\|_\infty .$$

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Our main result (informal)

For random translation-invariant (TI) MPS and PEPS with periodic boundary conditions this intuition is generically true, with a short correlation length.

→ going to 0 as d, q grow

→ with probability going to 1 (exponentially) as d, q grow

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Two ways of proving this: (suited to different dimensional regimes)

- Show that the *transfer operator* associated to the random TNS generically has a *large (upper) spectral gap*. going to 1 as d, q grow
- Show that the *parent Hamiltonian* of the random TNS generically has a *large (lower) spectral gap*. going to 1 as d, q grow

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Correlation length in a TI TNS and spectrum of its transfer operator

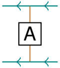
$|\varphi\rangle \in (\mathbf{C}^d)^{\otimes MN}$ a TI MPS ($M=1$) or PEPS ($M>1$) with transfer operator T on $(\mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$.
A, B quasi-local observables, i.e. on 1 subsystem $\mathbf{C}^d$ or 1 column of M subsystems $(\mathbf{C}^d)^{\otimes M}$, separated by L sites or columns.

→ do not depend on the sites, by TI of $|\varphi\rangle$

Reading the diagrams of $v_\varphi(A,B,L)$, $v_\varphi(A)$, $v_\varphi(B)$ 'horizontally' instead of 'vertically', we get:

$$\gamma_\varphi(A,B,L) = \left|\mathrm{Tr}\left(\tilde{A}T^L\tilde{B}T^{N-L-2}\right) - \mathrm{Tr}\left(\tilde{A}T^{N-1}\right)\mathrm{Tr}\left(\tilde{B}T^{N-1}\right)\right|.$$

$A : \mathbf{C}^d \to \mathbf{C}^d$



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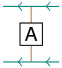

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Important consequence:
Denote by $\lambda_1(T)$ and $\lambda_2(T)$ the largest and second largest (in modulus) eigenvalues of T.
There exists $C(T) < \infty$ such that, setting $\varepsilon(T) = |\lambda_2(T)|/|\lambda_1(T)|$, we have

$$\gamma_\varphi(A,B,L) \leqslant C(T)\varepsilon(T)^L \|A\|_\infty \|B\|_\infty.$$

$\longrightarrow$ Correlations between two 1-site or 1-column observables decay exponentially with the distance separating the two sites or columns, at a rate $|\log \varepsilon(T)|$, i.e. $\xi(\varphi) = 1/|\log \varepsilon(T)|$.

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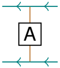
Reading the diagrams of $v_\varphi(A,B,L)$, $v_\varphi(A)$, $v_\varphi(B)$ 'horizontally' instead of 'vertically', we get:

$$\gamma_\varphi(A,B,L) = \left| \mathrm{Tr}\left(\tilde{A}T^L\tilde{B}T^{N-L-2}\right) - \mathrm{Tr}\left(\tilde{A}T^{N-1}\right)\mathrm{Tr}\left(\tilde{B}T^{N-1}\right) \right|.$$

$A : \mathbf{C}^d \to \mathbf{C}^d$



$\tilde{A} : \mathbf{C}^q \otimes \mathbf{C}^q \to \mathbf{C}^q \otimes \mathbf{C}^q$

Important consequence:
Denote by $\lambda_1(T)$ and $\lambda_2(T)$ the largest and second largest (in modulus) eigenvalues of T.
There exists $C(T) < \infty$ such that, setting $\varepsilon(T) = |\lambda_2(T)|/|\lambda_1(T)|$, we have

$$\gamma_\varphi(A,B,L) \leqslant C(T)\varepsilon(T)^L \|A\|_\infty \|B\|_\infty.$$

$\longrightarrow$ Correlations between two 1-site or 1-column observables decay exponentially with the distance separating the two sites or columns, at a rate $|\log\varepsilon(T)|$, i.e. $\xi(\varphi) = 1/|\log\varepsilon(T)|$.

Question: For a random transfer operator T what are the typical values of $|\lambda_1(T)|$ and $|\lambda_2(T)|$?

Typical spectral gap of the transfer operator of a random MPS

Theorem [Typical spectral gap of the random MPS transfer operator T]

There exist constants $C < \infty$, $c > 0$ such that, for any $d, q \in \mathbf{N}$,

$$\mathbf{P}\left(|\lambda_1(T)| \geqslant 1 - \frac{C}{\sqrt{d}} \text{ and } |\lambda_2(T)| \leqslant \frac{C}{\sqrt{d}} \right) \geqslant 1 - e^{-cq}.$$

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Corollary [Typical correlation length in the random MPS $|\varphi_N\rangle$]

There exist constants $C' < \infty$, $c > 0$ such that, for any $d, q, N \in \mathbf{N}$,

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Proof idea: Approximate first eigenvector for T: maximally entangled state $|\psi\rangle \in \mathbf{C}^q \otimes \mathbf{C}^q$.
By minimax principle: $|\lambda_1(T)| \geqslant |\langle\psi|T|\psi\rangle|$ and $|\lambda_2(T)| \leqslant \|T(\mathrm{I} - |\psi\rangle\langle\psi|)\|_\infty$.
↳ doing as if T were Hermitian...

- $\mathbf{E}|\langle\psi|T|\psi\rangle| = 1$ and $\mathbf{E}\|T(\mathrm{I} - |\psi\rangle\langle\psi|)\|_\infty \leqslant C/\sqrt{d}$ (Gaussian moment computations).
- For both quantities, small probability of deviating from average (Gaussian concentration).

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Remark: Such typical scaling for $|\lambda_1(T)|, |\lambda_2(T)|$ remains true for various models of random T: independent (potentially sparse) matrices with arbitrary independent entries (Lancien/Youssef), independent Haar (or even just 2-design) unitaries (Hastings, Pisier, Timhadjelt, Lancien).

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Main issue: Results are valid only in the regime where d, q grow polynomially with M.
$\longrightarrow$ Enforce this scaling by *blocking*: Redefine 1 site as being a sublattice of $\log M$ sites.

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2. Random MPS and PEPS: construction and statement of the main questions & results
3. Typical correlation length in a random TNS through typical spectral gap of its transfer operator
4. Typical correlation length in a random TNS through typical spectral gap of its parent Hamiltonian
5. Open questions and perspectives

Parent Hamiltonian of a TNS

Parent Hamiltonian of a TNS: local Hamiltonian which has the TNS as ground state.

In the *injectivity regime*, the latter is unique (Cirac/Pérez-García/Verstraete/Wolf).

↳ the 1-site tensor, viewed as a map from bond space to physical space, is injective

MPS

$$\Phi : (\mathbf{C}^q)^{\otimes 2} \to \mathbf{C}^d$$

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Fact: If the parent Hamiltonian of a TNS is gapped above its ground state energy then the correlations between two local observables decay exponentially with the distance separating the two sites (or subregions of sites). This follows from the *Lieb-Robinson bound* (Hastings/Koma...) or the *detectability lemma* (Aharonov/Arad/Landau/Vazirani...).

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Parent Hamiltonian of our random TNS: Set $d_i = q^2$ for MPS and $d_i = q^4$ for PEPS.
The random 1-site tensor Φ is almost surely injective for $d \geqslant d_i$.
In this regime, the parent Hamiltonian H is a 2-local frustration-free random Hamiltonian whose unique ground state is the random TNS.

↳ the global minimizer minimizes each term
↳ each term acts non-trivially on 2 neighboring sites

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There exist constants $C < \infty$, $c > 0$ such that, for any $N \in \mathbf{N}$ and any $d, q \in \mathbf{N}$ satisfying $d \gg d_i$,

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Remark: Such results seem to remain true for various models of random MPS and PEPS (in progress with Le Nestour/Pérez-García).

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- What about other models of random TNS? For instance:
  - Different distribution of the random 1-site tensor (e.g. invariant under some group action).
    $\longrightarrow$ Typical properties of TNS with local symmetry (in progress with Le Nestour/Pérez-García).
  - Different geometry of the graph (e.g. higher-dimensional regular lattice or tree-like).
    $\longrightarrow$ AdS/CFT correspondence: random TNS on hyperbolic graphs as toy-models in holography (Hayden/Nezami/Qi/Thomas/Walter/Yang, Cheng/Lancien/Penington/Walter/Witteveen).

Perspective: what about estimating the typical amount of entanglement in TNS?

Observation: The amount of bipartite entanglement in a TNS can be upper bounded in terms of the bond dimension. Indeed, for any subregion having α boundary edges and V bulk vertices, its *entanglement entropy* is at most $\alpha \log q$, which is usually much smaller than $V \log d$.

↳ entropy of reduced state

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Fact: $E(\chi) = 0$ if and only if $|\chi\rangle$ is separable. And we always have $E(\chi) \leqslant (N-1)\log d$.

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Question: What is the generic amount of GME in a random TNS $|\varphi\rangle \in (\mathbf{C}^d)^{\otimes N}$?

MPS case: $E(\varphi)$ is typically of order $(N-1)\log \min(d,q)$ (Fitter/Lancien/Nechita + in progress).

↳ not maximal if $q \ll d$ but scales extensively with N

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