Should entanglement measures be monogamous or faithful?

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1 Introduction

Seminal observation: A key feature of entanglement is that it cannot be shared unconditionally across many subsystems of a composite system [9].

→ **Sharpest manifestation:** If two parties A and B share a maximally entangled state, then they cannot share any correlation (even classical ones) with a third party C.

→ **More realistic scenario:** If A and B share a partially entangled (mixed) quantum state ρ_{AB}, then they may share part of this entanglement with other parties, but restrictions remain.
For instance: A state ρ_{AB} on $\mathrm{A}\otimes\mathrm{B}$ is called k-extendible if there exists a state ρ_{AB^k} on $\mathrm{A}\otimes\mathrm{B}^{\otimes k}$ which is invariant under permutation of the B subsystsems and s.t. $\mathrm{Tr}_{\mathrm{B}^{k-1}}\rho_{\mathrm{AB}^k}=\rho_{\mathrm{AB}}$.
If ρ_{AB} is infinitely-extendible, then it is separable [1, 4].

Question: How to formalize quantitatively this observation that entanglement is "monogamous"?

Natural idea: Given an entanglement measure E, show that, for any state ρ_{ABC},

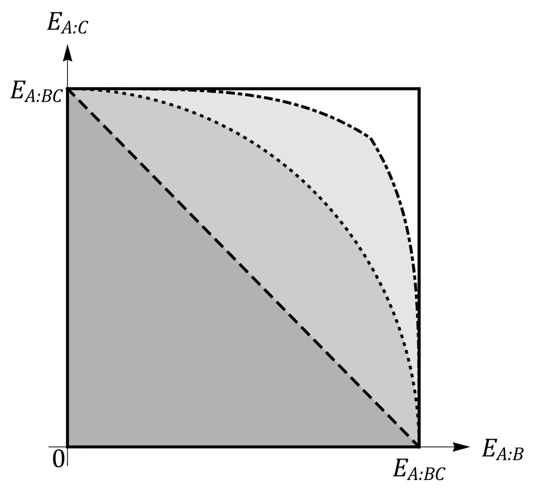
$$E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}}) \geqslant E_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) + E_{\mathrm{A:C}}(\rho_{\mathrm{AC}}). \qquad \text{(CKW)}$$

Inequality (CKW) holds true for the squared concurrence [3] or the squashed entanglement [2], but fails for many other entanglement measures.
Introducing rescalings or mixing different entanglement measures may allow to recover inequalities of this type, but what about a less ad hoc treatment?

Question: How to define monogamy of an entanglement measure in the most general possible terms?

Definition 1.1 (Generalized universal monogamy relation)**.**
An entanglement measure E is monogamous if there exists a non-trivial function $f:\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$ s.t., for any state ρ_{ABC} on any tripartite system $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}$,

$$E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}}) \geqslant f\big(E_{\mathrm{A:B}}(\rho_{\mathrm{AB}}), E_{\mathrm{A:C}}(\rho_{\mathrm{AC}})\big).$$



E is an entanglement monotone so w.l.o.g. $f(x,y)\geqslant\max(x,y)$.
Non-trivial constraint: $f(x,y)>\max(x,y)$ for at least some x,y.
→ If the only possible choice is $f(x,y)=\max(x,y)$, E drastically fails monogamy: knowing $z=E_{\mathrm{A:BC}}>0$ does not constrain in any way $x=E_{\mathrm{A:B}}$ and $y=E_{\mathrm{A:C}}$ in the interval $[0,z]$.
Intuition: E is monogamous if it obeys some trade-off between the values of $E_{\mathrm{A:B}}$ and $E_{\mathrm{A:C}}$ for a given $E_{\mathrm{A:BC}}$.
Form (CKW) of a monogamy relation: $f(x,y)=x+y$

2 Non-monogamy for the entanglement of formation and the relative entropy of entanglement

$\mathbf{E_F}$ (entanglement of formation) and $\mathbf{E_R}$ (relative entropy of entanglement):
Entropy: $S(\rho)=-\mathrm{Tr}(\rho\log\rho)$. Relative entropy: $D(\rho\|\sigma)=-\mathrm{Tr}(\rho[\log\rho-\log\sigma])$.
- $E_F(\rho_{\mathrm{A:B}}) := \inf\Big\{\sum_i p_i S\big(\mathrm{Tr}_{\mathrm{B}}|\psi_i\rangle\langle\psi_i|_{\mathrm{AB}}\big) \ : \ \sum_i p_i|\psi_i\rangle\langle\psi_i|_{\mathrm{AB}}=\rho_{\mathrm{AB}}\Big\}$.
- $E_R(\rho_{\mathrm{A:B}}) := \inf\Big\{D\big(\rho_{\mathrm{AB}}\|\sigma_{\mathrm{AB}}\big) \ : \ \sigma_{\mathrm{AB}} \text{ separable}\Big\}$.

Random induced states: System of interest H. Environment E.
Random mixed state model on H: $\rho=\mathrm{Tr}_{\mathrm{E}}|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a uniformly distributed pure state on $\mathrm{H}\otimes\mathrm{E}$.
Note: If $|\mathrm{E}|\leqslant|\mathrm{H}|$, ρ is uniformly distributed on the set of states on H with rank at most $|\mathrm{E}|$.

Theorem 2.1 (Generic non-monogamy for E_F and E_R)**.**
Let ρ_{ABC} be a random state on $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}\equiv\mathbf{C}^d\otimes\mathbf{C}^d\otimes\mathbf{C}^d$, induced by some environment $\mathrm{E}\equiv\mathbf{C}^s$, with $s\simeq\log d$. Then,
- *$E_F(\rho_{\mathrm{A:BC}})\leqslant\log d$ and $E_R(\rho_{\mathrm{A:BC}})\leqslant\log d$.*
- *With probability going to 1 (exponentially) as d grows, $E_F(\rho_{\mathrm{A:B}}),E_F(\rho_{\mathrm{A:C}})=(1-o(1))\log d$ and $E_R(\rho_{\mathrm{A:B}}),E_R(\rho_{\mathrm{A:C}})=(1-o(1))\log d$.*

Remark: Any value z for $E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}})$ is indeed attainable, on systems of local dimension 2^z.

Needed technical results in the proof: Typical value of E_F and E_R for random induced states (cf. also [6]).
There exist universal constants $C,c,c'>0$ s.t., for ρ_{AB} a random state on $\mathrm{A}\otimes\mathrm{B}\equiv\mathbf{C}^d\otimes\mathbf{C}^d$, induced by some environment $\mathrm{E}\equiv\mathbf{C}^s$, with $Cd\leqslant s\leqslant d^2$, we have

$$\forall\, t>0,\ \mathbf{P}\left(\left|E_F(\rho_{\mathrm{AB}})-\log d+\frac{1}{2\ln 2}\right|>t\right)\leqslant e^{-cd^2t^2/\log^2 d} \text{ and } \mathbf{P}\left(\left|E_R(\rho_{\mathrm{AB}})-\log\frac{d^2}{s}\right|>t\right)\leqslant e^{-c'st^2}.$$

Conclusion: E_F and E_R are non-monogamous, in the most general sense. This feature even becomes generic for high-dimensional quantum states.

Question: Is this just a consequence of their subadditivity [5, 10]?

3 Non-monogamy for a whole class of additive entanglement measures

Requirements on the considered entanglement measure E:
(1) **Normalization:** For any state ρ_{AB}, $E_{\mathrm{A:B}}(\rho_{\mathrm{AB}})\leqslant\min(\log|\mathrm{A}|,\log|\mathrm{B}|)$.
(2) **Lower boundedness on the anti-symmetric state α:** There exist universal constants $c,t>0$ s.t. $E_{\mathrm{A:A'}}(\alpha_{\mathrm{AA'}})\geqslant c/\log^t|\mathrm{A}|$.
(3) **Additivity under tensor product:** For any state ρ_{AB}, $E_{\mathrm{A}^m:\mathrm{B}^m}(\rho_{\mathrm{AB}}^{\otimes m})=m\,E_{\mathrm{A:B}}(\rho_{\mathrm{AB}})$.
(4) **Linearity under locally orthogonal mixture:** For any states $\rho_{\mathrm{AB}},\sigma_{\mathrm{AB}}$ s.t. $\mathrm{Tr}(\rho_{\mathrm{A}}\sigma_{\mathrm{A}})=\mathrm{Tr}(\rho_{\mathrm{B}}\sigma_{\mathrm{B}})=0$, $E_{\mathrm{A:B}}(\lambda\rho_{\mathrm{AB}}+(1-\lambda)\sigma_{\mathrm{AB}})=\lambda E_{\mathrm{A:B}}(\rho_{\mathrm{AB}})+(1-\lambda)E_{\mathrm{A:B}}(\sigma_{\mathrm{AB}})$.

Remarks on these requirements:
- Assumption (3) holds by construction for any regularized entanglement measure, i.e. one defined as $E^{\infty}_{\mathrm{A:B}}(\rho_{\mathrm{AB}}):=\lim_{n\to+\infty}\frac{1}{n}E_{\mathrm{A}^n:\mathrm{B}^n}(\rho_{\mathrm{AB}}^{\otimes n})$.
- Assumption (2) is a faithfulness (or geometry-preserving) property: in 1-norm distance α is dimension-independently separated from the set of separable states, so an entanglement measure which faithfully captures this geometrical feature should be dimension-independently (or weakly dimension-dependently) bounded away from 0 on α.

Examples of entanglement measures fulfilling these requirements:
E_F^{∞} and E_R^{∞}, the regularized versions of E_F and E_R.

Theorem 3.1 (Non-monogamy for any E satisfying requirements (1–4))**.**
There exists a state ρ_{ABC} on $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}\equiv\mathbf{C}^d\otimes(\mathbf{C}^d)^{\otimes 2^k}\otimes(\mathbf{C}^d)^{\otimes 2^k}$, where $0\leqslant k\leqslant\lfloor\log d\rfloor$, s.t.

$$E_{\mathrm{A:B}}(\rho_{\mathrm{AB}}),\ E_{\mathrm{A:C}}(\rho_{\mathrm{AC}})\geqslant(1-o(1))\,E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}}) \text{ as } d\to+\infty.$$

Remark: By considering tensor products and mixtures, any value z for $E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}})$ is indeed attainable, on systems of suitably large local dimensions.

Needed observation in the proof: The fully anti-symmetric state α_{A^n} on $\mathrm{A}^{\otimes n}$ is s.t., for any $m\leqslant n$, $\mathrm{Tr}_{\mathrm{A}^{n-m}}\alpha_{\mathrm{A}^n}=\alpha_{\mathrm{A}^m}$ is the fully anti-symmetric state on $\mathrm{A}^{\otimes m}$

Conclusion: Additive entanglement measures may also be non-monogamous, in the most general sense, as soon as they are strongly faithful. Explicit construction of a counter-example, based on the anti-symmetric state.

Question: Can monogamy still be rescued in some way?

4 Recovering monogamy with non-universal relations

Definition 4.1 (Generalized non-universal monogamy relation)**.**
An entanglement measure E is monogamous if, given a tripartite system $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}$, there exists a non-trivial function $f_{\mathrm{A,B,C}}:\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$ s.t., for any state ρ_{ABC} on $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}$,

$$E_{\mathrm{A:BC}}(\rho_{\mathrm{ABC}})\geqslant f_{\mathrm{A,B,C}}\big(E_{\mathrm{A:B}}(\rho_{\mathrm{AB}}),E_{\mathrm{A:C}}(\rho_{\mathrm{AC}})\big).$$

Theorem 4.2 (Dimension-dependent monogamy relations for E_F and E_R^{∞})**.**
There exist universal constants $c,c'>0$ s.t., for any state ρ_{ABC} on $\mathrm{A}\otimes\mathrm{B}\otimes\mathrm{C}\equiv\mathbf{C}^d\otimes\mathbf{C}^d\otimes\mathbf{C}^d$,

$$E_F(\rho_{\mathrm{A:BC}})\geqslant\max\left(E_F(\rho_{\mathrm{A:B}})+\frac{c}{d^2\log^8 d}E_F(\rho_{\mathrm{A:C}})^8,\ E_F(\rho_{\mathrm{A:C}})+\frac{c}{d^2\log^8 d}E_F(\rho_{\mathrm{A:B}})^8\right),$$
$$E_R^{\infty}(\rho_{\mathrm{A:BC}})\geqslant\max\left(E_R^{\infty}(\rho_{\mathrm{A:B}})+\frac{c'}{d^2\log^4 d}E_R^{\infty}(\rho_{\mathrm{A:C}})^4,\ E_R^{\infty}(\rho_{\mathrm{A:C}})+\frac{c'}{d^2\log^4 d}E_R^{\infty}(\rho_{\mathrm{A:B}})^4\right).$$

Remark: Most likely, neither the dimensional pre-factors nor the powers 8 and 4 appearing in these relations are tight.

Needed ingredients in the proof: Hybrid monogamy-type relations involving filtered through local measurements entanglement measures (cf. also [8]).

Conclusion: In any fixed finite dimension, E_F and E_R^{∞} can be regarded as monogamous. But the monogamy relations that they obey become trivial in the limit of infinite dimension.

5 Concluding remarks and perspectives

- What happens when more than 3 parties are involved? More precisely, what are the conceptual limitations and what would be the practical implications in such many-body scenario (e.g. in quantum condensed-matter physics or gravity)?

- What about hybrid monogamy-type relations mixing different entanglement measures (and/or even involving measures of other kinds of correlations)?

- Can we understand better the connections of this entanglement monogamy phenomenon with the quantum marginal problem and the question of quantum information distribution/recoverability...?

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