# de Finetti reductions for partially exchangeable distributions

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## The finite de Finetti theorem

Let $V$ be a finite alphabet, $|V| = d$, and consider probability measures on $V^n$ which are *symmetric under permutations*:

$$\forall \sigma \in \mathcal{S}_n, \qquad \mathbb{P}[x_1, x_2, \ldots, x_n] = \mathbb{P}[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}].$$

Such probability distributions are called *exchangeable*. In particular, i.i.d. distributions are exchangeable

$$\mathbb{P} = \pi^{\otimes n} \quad \text{i.e.} \quad \mathbb{P}[x_1, x_2, \ldots, x_n] = \prod_{i=1}^{n} \pi(x_i) = \prod_{a \in V} \pi(a)^{|x^{-1}(a)|}.$$

**Theorem([1]).** Let $\mathbb{P}$ be an exchangeable probability distribution on $V^n$. Then, for $k \ll n$, its $k$-marginal $\mathbb{P}_k$ is close to a convex mixture of i.i.d. distributions. More precisely, for any $k \leq n$, there exists a probability measure $\mu$ on $\mathcal{P}(V)$ such that

$$\left\| \mathbb{P}_k - \int \pi^{\otimes k} \mathrm{d}\mu(\pi) \right\|_{\mathrm{TV}} \leq \frac{2kd}{n}.$$

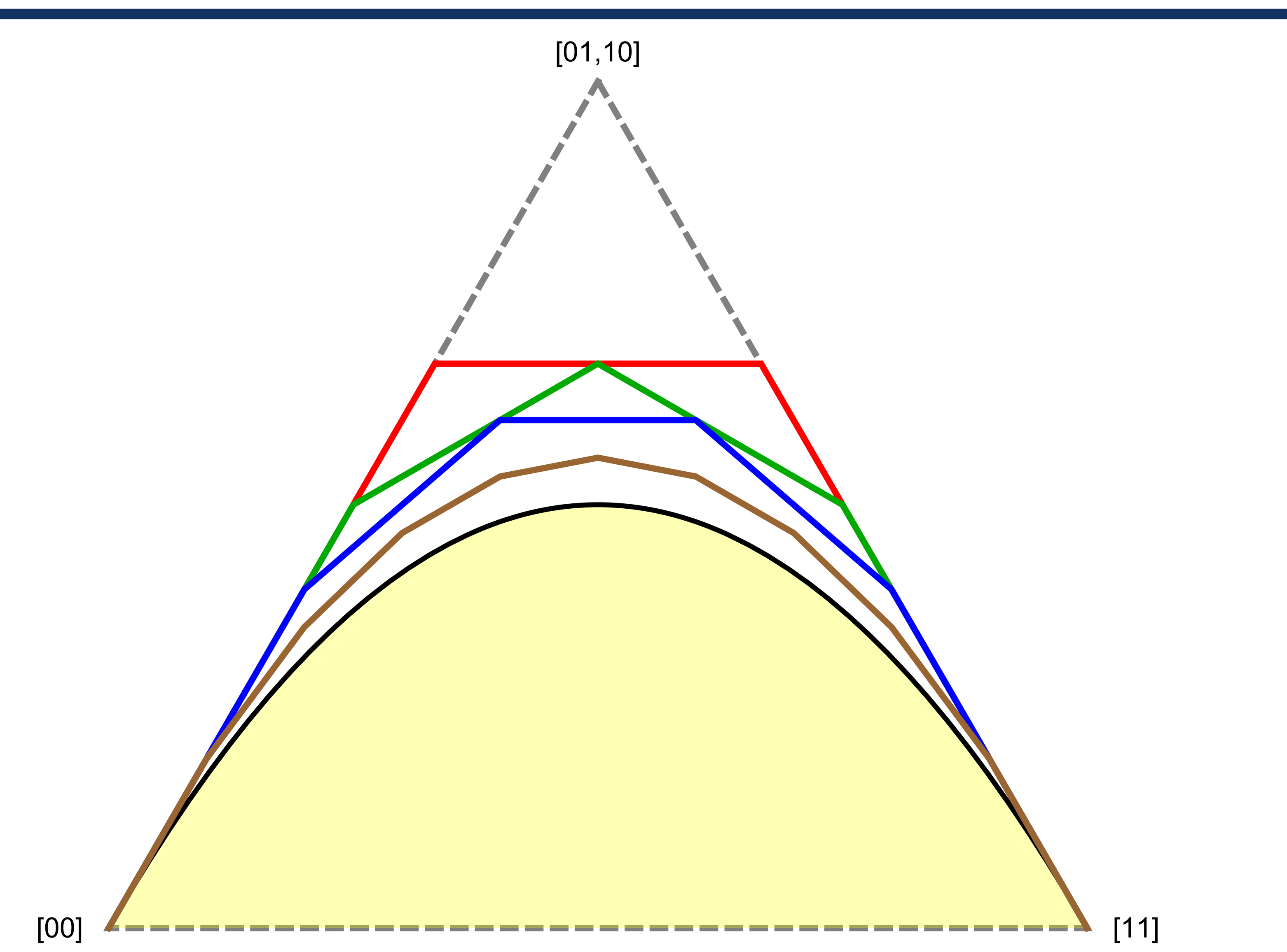


Figure 1: The filled yellow area corresponds to mixtures of i.i.d. distributions on $\{0,1\}^2$. The lines delimit $k = 2$-marginals of exchangeable distributions on $\{0,1\}^n$, with $n = 3, 4, 5, 10$.

## de Finetti reductions

Let $\sim$ be an equivalence relation on $V^n$, and denote by $\mathcal{P}_\sim(V^n)$ the convex set of $\sim$ invariant distributions:

$$\mathbb{P} \in \mathcal{P}_\sim(V^n) \iff \forall x \sim y \in V^n, \quad \mathbb{P}[x] = \mathbb{P}[y].$$

The set $\mathcal{P}_\sim(V^n)$ is a simplex, whose extreme points are the uniform distributions on the equivalence classes of $\sim$. Let $\Pi_n \subseteq \mathcal{P}_\sim(V^n)$ be a distinguished subclass of $\sim$ exchangeable distributions.

**Definition.** We say that the pair $(\sim, \Pi)$ admits a *flexible de Finetti reduction* if, for any probability distribution $\mathbb{P} \in \mathcal{P}_\sim(V^n)$, we have, point-wise,

$$\mathbb{P} \leq \mathrm{poly}(n) \int_{\pi \in \Pi_n} F(\mathbb{P}, \pi)^2 \pi \mathrm{d}\nu(\pi),$$

where $F$ is the fidelity, $\mathrm{poly}(n)$ is a polynomial in $n$ and $\nu$ is a probability distribution on $\Pi(V^n)$.

## Three examples

- *Exchangeability*, with $\Pi_n = \{\pi^{\otimes n} : \pi \in \mathcal{P}(V)\}$
- *Markov exchangeability [2]:* if $x, y \in V^n$, define $x \sim y$ iff $x_1 = y_1$ and, for all $a, b \in V$, $t_{ab}(x) = t_{ab}(y)$, where
$$t_{ab}(x) = |\{i \in [1, n-1] : (x_i, x_{i+1}) = (a, b)\}|$$
The class of distinguished measures $\Pi_n = \{\mathbb{Q}_{a,M}\}$ is indexed by couples $(a, M)$, where $a \in V$ and $M$ is a Markov matrix
$$\mathbb{Q}_{a,M}[x_1, \ldots x_n] = \mathbf{1}_{x_1 = a} \prod_{i,j \in V} M_{ij}^{t_{ij}(x)}$$
- *$\ell$-Markov exchangeability:* $x \sim y$ iff $x_i = y_i$ for $i = 1, \ldots, \ell$ and, for all $a = (a_1, \ldots, a_{\ell+1}) \in V^{\ell+1}$, $t_a(x) = t_a(y)$, where
$$t_a(x) = |\{\text{occurrences of the sequence } a_1, .., a_{\ell+1} \text{ in } x\}|$$

One can also consider *double partial exchangeability*, where $V = V_1 \times V_2$ is equipped with the Cartesian product of two equivalence relations on $V_{1,2}$.

## Our main result

The three examples mentioned above (exchangeability, Markov exchangeability, and $\ell$-Markov exchangeability), together with the appropriate classes of distributions, admit **flexible de Finetti reductions** with polynomial pre-factors of respective degrees

(EXCH): $2(d-1)$ (M-EXCH): $d(2d+1)-1$ ($\ell$-M-EXCH): $d^\ell(2d+1)-1$.

## Tools: the BEST theorem

Our results follow from estimates of the size of the equivalence classes on $V^n$. Inspired by [6, 7], we construct a bijection between the elements of a given equivalence class and the *Eulerian cycles* of a (class-dependent) graph.

**Theorem([3, 4]).** Consider an Eulerian directed multigraph $G$ with a marked edge $e_0 \in E$ and a marked vertex $w_0 \in V$. Let $T(G, w_0)$ denote the number of spanning trees of $G$ oriented towards the vertex $w_0$ (i.e. all orientations in the tree are pointing towards $w_0$). Then, the number of Eulerian cycles of $G$ starting with the edge $e_0$ is given by

$$T(G, w_0) \prod_{i \in V} (\mathrm{outdeg}(i) - 1)!.$$

Remarkably, $T(G, w_0)$ is independent of the choice of the marked vertex $w_0$.

**Example.** Let $x = (11323122) \in \{1,2,3\}^8$ and consider $\mathcal{C}$, its equivalence class w.r.t. Markov exchangeability. The class $\mathcal{C}$ has 12 elements, and $T(G) = 3$.

$$t = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

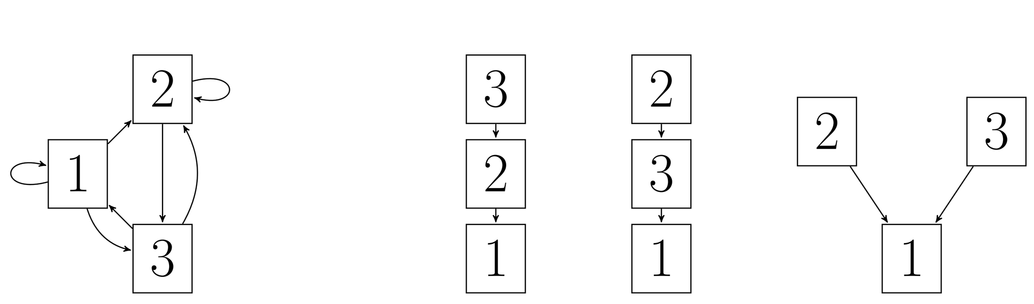


Figure 2: The matrix $t$ and the graph $G$ associated to $x = (11323122)$, as well as the three oriented trees flowing towards the vertex 1.

## Conditional distributions

**Theorem.** For any [EXCH/M-EXCH/$\ell$-M-EXCH]-exchangeable probability distribution $\mathbb{P} \in \mathcal{P}_\sim(V^n)$ with $V = A \times X$, we have, point-wise,

$$\mathbb{P}_{A^n|X^n} \leq \mathrm{poly}(n) \int_{\pi \in \Pi_n(A^n \times X^n)} \pi_{A^n|X^n} \, d\nu(\pi),$$

where $\mathrm{poly}(n)$ is a polynomial in $n$ and $\nu$ is a probability distribution on $\Pi_n(A^n \times X^n)$.

## References

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