(In)compatibility of random measurements

Based on the work arXiv:2507.20600, with Andreas Bluhm and Ion Nechita

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Plan

positive-operator-valued measure

Measurement (aka *POVM*) with k outcomes: $\mathrm{M} = (M_i)_{i\in[k]}$ with $M_1,\ldots,M_k \geqslant 0$ s.t. $\sum_{i=1}^{k} M_i = I$.

measurement effects

If all M_i's are projectors, M is called a *PVM*.

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↱ positive-operator-valued measure (POVM)

↳ measurement effects (M_i)

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POVMs are compatible if they can be seen as marginals of a joint POVM (or equivalently if their outcome statistics can be reproduced by post-processing the outcomes of a single POVM).

Definition [Compatible POVMs]

Let $\mathrm{M}_1 = (M_{i_1|1})_{i_1\in[k_1]},\ldots,\mathrm{M}_n = (M_{i_n|n})_{i_n\in[k_n]}$ be n POVMs on $\mathbf{C}^d$. We say that they are *compatible* if there exists a POVM $\mathrm{L} = (L_{i_1,\ldots,i_n})_{i_1\in[k_1],\ldots,i_n\in[k_n]}$ on $\mathbf{C}^d$ s.t.

$$\forall\, x\in[n],\, i\in[k_x],\ M_{i|x} = \sum_{y\in[n],y\neq x}\ \sum_{i_y\in[k_y]} L_{i_1,\ldots,i_{x-1},i,i_{x+1},\ldots,i_n}.$$

Remark: A sufficient condition for compatibility is pairwise commutativity of measurement effects. For PVMs, this is actually a necessary and sufficient condition.

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Interest: Incompatibility is a *resource* for several quantum information processing tasks (e.g. required for the violation of Bell or steering inequalities).
$\longrightarrow$ In practice, need for quantifying it.

Fact: All POVMs can be made compatible by adding enough noise to measurement effects.
$\longrightarrow$ To quantify incompatibility of POVMs: minimal amount of noise that makes them compatible.

Given $\mathrm{M} = (M_i)_{i\in[k]}$ a POVM and $t \in [0,1]$, define the POVM $\mathrm{M}^{(t)} := \left(tM_i + (1-t)\frac{I}{k}\right)_{i\in[k]}$.
↳ (uniformly) noisy version of M

Definition [Compatibility degree of POVMs]

Let $\mathrm{M}_1,\ldots,\mathrm{M}_n$ be n POVMs on $\mathbf{C}^d$. Their *compatibility degree* is defined as

$$\tau(\mathrm{M}_1,\ldots,\mathrm{M}_n) := \max\left\{ t\in[0,1] : \mathrm{M}_1^{(t)},\ldots,\mathrm{M}_n^{(t)} \text{ are compatible POVMs} \right\}.$$

Quantifying incompatibility of quantum measurements

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Question: For a given underlying dimension d, what is the minimal amount of noise that makes all sets of n POVMs with some fixed number of outcomes compatible?

Definition [Minimal compatibility degree]

The *minimal compatibility degree* for n POVMs on $\mathbf{C}^d$ with $k_1, \ldots, k_n$ outcomes is defined as

$$\tau(d, n, (k_1, \ldots, k_n)) := \min_{\mathrm{M}_1, \ldots, \mathrm{M}_n} \tau(\mathrm{M}_1, \ldots, \mathrm{M}_n),$$

where the minimum is taken over all n POVMs $\mathrm{M}_1 = (M_{i_1|1})_{i_1\in[k_1]}, \ldots, \mathrm{M}_n = (M_{i_n|n})_{i_n\in[k_n]}$ on $\mathbf{C}^d$.

Previously known results and our goal

Some previously known upper and lower bounds on the minimal compatibility degree:

- $\tau(d,n,(2,\ldots,2)) \geqslant \frac{1}{\sqrt{n}}$, with equality if $n = O(\log(d))$ (Bluhm/Nechita).
- $\frac{n+d}{n(d+1)} \leqslant \tau(d,n,(d,\ldots,d)) \leqslant \frac{n+\sqrt{d}}{n(\sqrt{d}+1)}$, with equality in the upper bound if $n=2$ (Heinosaari/Schultz/Toigo/Ziman, Brunner/Designolle/Fröwis/Skrzypczyk).
- $\tau(d,n,(k_1,\ldots,k_n)) \geqslant \frac{n+kd}{n(kd+1)}$, where $k = \max(k_1,\ldots,k_n)$ (Bluhm/Nechita).

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Question: What about estimating the *typical compatibility degree*, rather than the minimal one?
$\longrightarrow$ How incompatible are POVMs sampled at random (inside a given subclass of POVMs)?

Remark: One may get tighter upper bounds on the minimal compatibility degree as a by-product (if random POVMs turn out to be very incompatible).

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Approach:

1. Develop *general techniques* to prove incompatibility (e.g. incompatibility witnesses).
2. Apply them to *random instances*, with tools from *random matrix theory* and *free probability*.

Results: Both *asymptotic* and *non-asymptotic*.
- *asymptotic* → almost sure behavior, in the limit of infinite dimension
- *non-asymptotic* → high probability behavior, in large but finite dimension

Plan

(In)compatibility of 2 random dichotomic projective measurements

Given a subspace E of $\mathbf{C}^d$, let $\mathrm{P}_E := (P_E, I - P_E)$ be the associated dichotomic PVM.

[$E = U\hat{E}$ with $\begin{cases} \hat{E} = \mathrm{span}\{|1\rangle, \ldots, |d'\rangle\} \\ U \text{ Haar distributed unitary on } \mathbf{C}^d \end{cases}$ is a uniformly distributed d'-dimensional subspace of $\mathbf{C}^d$.]

Theorem [Compatibility degree of 2 random dichotomic PVMs]

Fix $\alpha, \beta \in]0,1[$ s.t. $\left(\alpha - \frac{1}{2}\right)^2 + \left(\beta - \frac{1}{2}\right)^2 \leqslant \frac{1}{4}$ and let $E, F \subset \mathbf{C}^d$ be independent uniformly distributed subspaces s.t. $\frac{\dim(E)}{d} \to \alpha$, $\frac{\dim(F)}{d} \to \beta$ as $d \to \infty$. Then,

$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\left| \tau(\mathrm{P}_E, \mathrm{P}_F) - \frac{1}{\sqrt{2}} \right| \leqslant \varepsilon \right) \underset{d\to\infty}{\longrightarrow} 1.$$

$\longrightarrow$ 2 random dichotomic PVMs on independent uniformly distributed subspaces of $\mathbf{C}^d$ with dimensions ‘close enough’ to $\frac{d}{2}$ are asymptotically maximally incompatible.

Remark: If the dimensions are ‘too far away’ from $\frac{d}{2}$, we have instead, say for $\alpha = \beta$,

$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\left| \tau(\mathrm{P}_E, \mathrm{P}_F) - \frac{1}{\sqrt{\lambda_\alpha} + \sqrt{1 - \lambda_\alpha}} \right| \leqslant \varepsilon \right) \underset{d\to\infty}{\longrightarrow} 1, \text{ where } \lambda_\alpha = 4\alpha(1-\alpha).$$

Proof idea for the upper bound

Lemma [Pauli compression criterion for incompatibility of dichotomic PVMs]

Let $P_1, \ldots, P_n$ be projectors on $\mathbf{C}^d$ and $\Sigma_1, \ldots, \Sigma_n$ be (generalized) Pauli observables on $\mathbf{C}^{d'}$.
Suppose that there exist an isometry $V : \mathbf{C}^{d'} \to \mathbf{C}^d$ and $\varepsilon > 0$ s.t. $\sum_{x=1}^{n} \left\| V^* P_x V - \frac{I+\Sigma_x}{2} \right\|_\infty \leqslant \frac{\varepsilon}{2}$.
Then, $\tau(\mathrm{P}_1, \ldots, \mathrm{P}_n) \leqslant \frac{1}{\sqrt{n} - \varepsilon}$.

This follows from three observations:

1. For any projectors $P_1, \ldots, P_n$ and isometry V, $\tau(\mathrm{P}_1, \ldots, \mathrm{P}_n) \leqslant \tau(V^*\mathrm{P}_1 V, \ldots, V^*\mathrm{P}_n V)$.
2. For any projectors $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$, $\left| \frac{1}{\tau(\mathrm{P}_1, \ldots, \mathrm{P}_n)} - \frac{1}{\tau(\mathrm{Q}_1, \ldots, \mathrm{Q}_n)} \right| \leqslant 2 \sum_{x=1}^{n} \|P_x - Q_x\|_\infty$.
3. For Pauli observables $\Sigma_1, \ldots, \Sigma_n$, $\tau\left(\left(\frac{I \pm \Sigma_1}{2} \right), \ldots, \left(\frac{I \pm \Sigma_n}{2} \right) \right) = \frac{1}{\sqrt{n}}$ (Bluhm/Nechita).

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Strategy: For 2 random projectors P_E, P_F on $\mathbf{C}^d$, we want to find a random isometry $V : \mathbf{C}^2 \to \mathbf{C}^d$ s.t. $V^* P_E V = \frac{I+\Sigma_Z}{2}$ and $V^* P_F V \simeq \frac{I+\Sigma_X}{2}$, i.e. random unit vectors $u \in E, u' \in E^\perp$ s.t. for all $v, w \in \{u, u'\}$, $\langle v|P_F|w\rangle \simeq \frac{1}{2}$, so that we can take $V = |u\rangle\langle 1| + |u'\rangle\langle 2|$.

[Principal angles between E and F: $0 \leqslant \theta_i \leqslant \frac{\pi}{2}$ s.t. $P_E = \sum_i |e_i\rangle\langle e_i|$, $P_F = \sum_j |f_j\rangle\langle f_j|$, $\langle e_i|f_j\rangle = \delta_{ij}\cos(\theta_i)$.]

Implementation: As $d \to \infty$, the distribution of principal angles between E and F converges to a well-identified distribution, whose support $[\theta_{\min}(\alpha,\beta), \theta_{\max}(\alpha,\beta)]$ contains $\frac{\pi}{4}$ (Aubrun).
So we just have to take u, u' as the corresponding eigenvectors in $E, E^\perp$.

Proof idea for the lower bound

Lemma [Jordan compatibility criterion for 2 noisy dichotomic PVMs]

Let P_1, P_2 be projectors on $\mathbf{C}^d$ and $t \in [0,1]$. Suppose that,
$\forall\ P_1' \in \{P_1, I - P_1\}, P_2' \in \{P_2, I - P_2\},\ t^2(P_1'P_2' + P_2'P_1') + t(1-t)(P_1' + P_2') + (1-t^2)\frac{I}{2} \geqslant 0.$
Then, $\mathrm{P}_1^{(t)}, \mathrm{P}_2^{(t)}$ are compatible, and thus $\tau(\mathrm{P}_1, \mathrm{P}_2) \geqslant t$.

This follows from *Jordan criterion*: Given POVMs $\mathrm{M} = (M_i)_{i\in[k]}, \mathrm{N} = (N_j)_{j\in[l]}$, if $M_iN_j + N_jM_i \geqslant 0$ for all $i \in [k], j \in [l]$, then M, N are compatible.

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Strategy: For 2 random projectors P_E, P_F on $\mathbf{C}^d$, we want to show that, asymptotically,
$\lambda_t(E,F) := \lambda_{\min}\left(t^2(P_EP_F + P_FP_E) + t(1-t)(P_E + P_F) + (1-t^2)\frac{I}{2}\right) \geqslant 0$ for $t < \frac{1}{\sqrt{2}}$.

Implementation: $\lambda_t(E,F) \geqslant \min\limits_{\theta}\left\{-t^2\cos(\theta)(1-\cos(\theta)) + t(1-t)(1-\cos(\theta)) + \frac{1-t^2}{2}\right\}$,
where the minimum is taken over all principal angles θ between E and F.
As $d \to \infty$, the support of the distribution of such angles contains $\arccos\left(\frac{1}{2t}\right)$ for $t < \frac{1}{\sqrt{2}}$ (Aubrun).
We thus have $\lambda_t(E,F) \geqslant \frac{1}{4} - \frac{t^2}{2} \geqslant 0$ for $t < \frac{1}{\sqrt{2}}$.

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Remark: This shows that Jordan criterion is asymptotically optimal for 2 random projectors.

Incompatibility of $n > 2$ random dichotomic projective measurements

In theory, the Pauli compression strategy could still work for $n > 2$ random dichotomic PVMs.
But in practice, exhibiting the random isometry that does the job is not that easy...
Issue: How to characterize the relative positions of $n > 2$ subspaces?

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Theorem [Compatibility degree of $n > 2$ random dichotomic PVMs]

Let $E_1, \ldots, E_n \subset \mathbf{C}^d$ be n independent uniformly distributed subspaces s.t. $\frac{\dim(E_x)}{d} \to \frac{1}{2}$ as $d \to \infty$ for each $x \in [n]$. Then,

$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\tau(\mathrm{P}_{E_1}, \ldots, \mathrm{P}_{E_n}) \leqslant \frac{2\sqrt{n-1}}{n} + \varepsilon \right) \underset{d\to\infty}{\longrightarrow} 1 .$$

$\longrightarrow$ For n large, n random dichotomic PVMs on independent uniformly distributed subspaces of $\mathbf{C}^d$ with dimensions $\frac{d}{2}$ have an asymptotic compatibility degree at most $\frac{2}{\sqrt{n}}$.

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Remark: We can also derive non-asymptotic quantitative estimates such as:

$$\forall\, n \leqslant C'd,\ \mathbf{P}\left(\tau(\mathrm{P}_{E_1}, \dots, \mathrm{P}_{E_n}) \leqslant \frac{C}{\sqrt{n}} \right) \geqslant 1 - e^{-cd},\ \text{for } C', C < \infty, c > 0 \text{ absolute constants.}$$

Consequence: For $n = O(d)$, $\tau(d, n, (2, \dots, 2)) = O\left(\frac{1}{\sqrt{n}}\right)$. [Previously known: for $n = O(\log d)$]

Proof idea

Lemma [Incompatibility witnesses for dichotomic PVMs]

Let $W_1, \dots, W_n$ be observables on $\mathbf{C}^d$.

Suppose that there exists a state ρ on $\mathbf{C}^d$ s.t., for all $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, $\sum_{x=1}^{n} \varepsilon_x W_x \leqslant \rho$.

Then, for all $\mathrm{P}_1, \dots, \mathrm{P}_n$ compatible dichotomic PVMs on $\mathbf{C}^d$, $\sum_{x=1}^{n} \mathrm{Tr}(W_x(2P_x - I)) \leqslant 1$.

This follows from expressing τ as the value of an SDP (Fernández/Pérez-García/Wolf, Bluhm/Nechita).

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Strategy: We take $W_x = \frac{s}{d} A_x := \frac{s}{d}(2P_x - I)$ for each $x \in [n]$ and $\rho = \frac{I}{d}$. If we can find s s.t. $\forall\, \varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, $\sum_{x=1}^{n} \varepsilon_x A_x \leqslant \frac{I}{s}$ and $\sum_{x=1}^{n} \mathrm{Tr}(A_x^2) > \frac{d}{st}$, then $\mathrm{P}_1^{(t)}, \dots, \mathrm{P}_n^{(t)}$ are not compatible.

[Given an observable M on $\mathbf{C}^d$, we denote by μ_M its spectral distribution, i.e. $\mu_M := \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i(M)}$.]

Implementation: For all $x \in [n]$ and $\varepsilon_x \in \{\pm 1\}$, $\mu_{\varepsilon_x A_x} \underset{d\to\infty}{\longrightarrow} \frac{1}{2}(\delta_{-1} + \delta_1) =: B$.
↳ Bernoulli distribution

So for all $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, $\mu_{\varepsilon_1 A_1 + \cdots + \varepsilon_n A_n} \underset{d\to\infty}{\longrightarrow} B^{\boxplus n}$ (by asymptotic freeness of $\varepsilon_1 A_1, \dots, \varepsilon_n A_n$).
↳ free additive convolution of n Bernoulli distributions

Since $\max \mathrm{supp}(B^{\boxplus n}) = 2\sqrt{n-1}$, for $s > s_* := \frac{1}{2\sqrt{n-1}}$, $\sum_{x=1}^{n} \varepsilon_x A_x \leqslant \frac{I}{s}$ asymptotically.

Since $A_x^2 = I$ for all $x \in [n]$, for $t > t_* := \frac{1}{s_* n} = \frac{2\sqrt{n-1}}{n}$, $\sum_{x=1}^{n} \mathrm{Tr}(A_x^2) > \frac{d}{s_* t}$.

Plan

Incompatibility of 2 random basis measurements

Given a unitary U on $\mathbf{C}^d$, let $\mathrm{B}_U := (U|1\rangle\langle 1|U^*, \ldots, U|d\rangle\langle d|U^*)$ be the associated basis PVM.

Theorem [Compatibility degree of 2 random basis PVMs]

Let U, V be independent Haar distributed unitaries on $\mathbf{C}^d$. Then,

$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\tau(\mathrm{B}_U, \mathrm{B}_V) \leqslant \frac{1}{2}\left(1 + (1+\varepsilon)\sqrt{\frac{3\log d}{d}}\right)\right) \underset{d\to\infty}{\longrightarrow} 1.$$

$\longrightarrow$ 2 random PVMs in independent uniformly distributed bases of $\mathbf{C}^d$ have an asymptotic compatibility degree at most $\frac{1}{2}$, with a finite-dimensional correction of order at most $\sqrt{\frac{\log d}{d}}$.

Given a unitary U on $\mathbf{C}^d$, let $\mathrm{B}_U := (U|1\rangle\langle 1|U^*, \ldots, U|d\rangle\langle d|U^*)$ be the associated basis PVM.

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[2 o.n.b. $\{u_1, \ldots, u_d\}, \{v_1, \ldots, v_d\}$ of $\mathbf{C}^d$ are *mutually unbiased bases (MUBs)* if, $\forall\, i,j \in [d]$, $|\langle u_i|v_j\rangle| = \frac{1}{\sqrt{d}}$.]

Comparison: For 2 basis PVMs, maximal incompatibility is achieved by those in MUBs.
For $\mathrm{B}_1, \mathrm{B}_2$ PVMs in MUBs: $\tau(\mathrm{B}_1, \mathrm{B}_2) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{d}+1}\right)$ (Carmeli/Heinosaari/Toigo).
$\longrightarrow$ Random bases are almost as incompatible.

Incompatibility of $n > 2$ random basis measurements

Theorem [Compatibility degree of $n > 2$ random basis PVMs]

Let $U_1, \ldots, U_n$ be n independent Haar distributed unitaries on $\mathbf{C}^d$. Then,

$$\mathbf{P}\left(\tau(\mathrm{B}_{U_1}, \ldots, \mathrm{B}_{U_n}) \leqslant \frac{C \log d}{n}\left(1 + \frac{n}{d}\right)\right) \geqslant 1 - e^{-c(n+d)\log d},$$

where $C < \infty, c > 0$ are absolute constants.

$\longrightarrow$ n random PVMs in independent uniformly distributed bases of $\mathbf{C}^d$ typically have a compatibility degree of order at most $\max\left(\frac{1}{n}, \frac{1}{d}\right) \log d$.

Consequence: For $n = \Omega(d)$, $\tau(d, n, (d, \ldots, d)) = \tilde{\Theta}\left(\frac{1}{d}\right)$. [Previously known: $\Omega(\frac{1}{d})$ and $O(\frac{1}{\sqrt{d}})$]

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Proof idea: Set $\eta(U_1, \dots, U_n) := \max\left\{\sum_{x=1}^{n} \|U_x \varphi\|_\infty^2 : \varphi \in \mathbf{C}^d, \|\varphi\|_2 = 1\right\}$.

By an incompatibility witness argument, $\mathrm{B}_{U_1}^{(t)}, \dots, \mathrm{B}_{U_n}^{(t)}$ are incompatible for $t > \frac{d \times \eta(U_1, \dots, U_n) - n}{n(d-1)}$.

So one just has to estimate $\eta(U_1, \dots, U_n)$ for n random unitaries $U_1, \dots, U_n$ on $\mathbf{C}^d$.

↳ asymptotic exact value for $n = 2$, non-asymptotic order of magnitude for $n > 2$

Plan

Summary and perspectives

↱ e.g. independent uniformly distributed dichotomic or basis PVMs

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- Key ingredient in many results: Make a guess for a potential *incompatibility witness*, which turns out to be generically close to optimal in high dimension.

  $\longrightarrow$ What about applying such strategy to the *generic detection of other non-classical features* (entanglement, extendibility, steering, etc)?

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