Quantum expanders – Random constructions & Applications

Based on the works:

• arXiv:1906.11682 (with David Pérez-Garcìa)
• arXiv:2302.07772 (with Pierre Youssef)
• arXiv:2409.17971

Cécilia Lancien

Institut Fourier Grenoble & CNRS

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Plan

Classical expanders

G a (directed or undirected) d-biregular graph on n vertices.

↳ d incoming and d outgoing edges at each vertex

A its (normalized) adjacency matrix, i.e. the $n \times n$ matrix s.t. $A_{kl} = e(l \to k)/d$ for all $1 \leqslant k, l \leqslant n$.

number of edges from vertex l to vertex k ↲

$\lambda_1(A), \ldots, \lambda_n(A)$ eigenvalues of A, ordered s.t. $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$.

G biregular $\Rightarrow$ A bistochastic $\Rightarrow$ $\lambda_1(A) = 1$, with associated eigenvector the uniform probability u.
The *spectral expansion parameter* of G is $\lambda(G) := |\lambda_2(A)|$. $(1/n, \ldots, 1/n) =$ ↲

Observation: $\lambda(G) = |\lambda_1(A - J)|$, where J is the adjacency matrix of the *complete graph* on n vertices, i.e. the matrix whose entries are all equal to $1/n$.
$\longrightarrow$ $\lambda(G)$ is a distance measure between G and the complete graph.

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Definition [Classical expander (informal)]

A d-biregular graph G on n vertices is an *expander* if it is sparse (i.e. $d \ll n$) and spectrally expanding (i.e. $\lambda(G) \ll 1$).

$\longrightarrow$ G is both 'economical' and 'resembling' the complete graph.
For instance, a random walk supported on G converges fast to equilibrium:
For any probability p on $\{1, \ldots, n\}$, $\forall\, q \in \mathbf{N}$, $\|A^q p - u\|_1 \leqslant \sqrt{n}\|A^q p - u\|_2 \leqslant \sqrt{n}\lambda(G)^q$.

exponential convergence, at rate $|\log \lambda(G)|$ ↲

Quantum analogue of the transition matrix associated to a biregular graph

Classical - Quantum correspondence:

- $p \in \mathbf{R}^n$ probability vector $\leftrightsquigarrow$ $\rho \in \mathcal{M}_n(\mathbf{C})$ density operator (aka *quantum state*).
  ↳ self-adjoint positive semidefinite trace 1 operator
- $A : \mathbf{R}^n \to \mathbf{R}^n$ transition matrix $\leftrightsquigarrow$ $\Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$ *quantum channel*.
  ↳ completely positive (CP) trace-preserving (TP) linear map
- G biregular: A leaves u invariant $\leftrightsquigarrow$ Φ unital: Φ leaves I/n invariant.
  ↳ maximally mixed density operator

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Question: What is the analogue of the degree in the quantum setting?

Answer: The Kraus rank.
Given a CP map Φ on $\mathcal{M}_n(\mathbf{C})$, a *Kraus representation* of Φ is of the form:

$$\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^{d} K_i X K_i^* \in \mathcal{M}_n(\mathbf{C}), \text{ where } K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C}). \quad (\star)$$

↳ Kraus operators of Φ

The minimal d s.t. Φ can be written as $(\star)$ is the *Kraus rank* of Φ (it is always at most n^2).
[Note: Φ is TP iff $\sum_{i=1}^d K_i^* K_i = I$. Φ is unital iff $\sum_{i=1}^d K_i K_i^* = I$.]

Indeed, the degree and the Kraus rank both quantify the 1-iteration spreading:

- G a degree d graph: If $|\mathrm{supp}(p)| = 1$, then $|\mathrm{supp}(Ap)| \leqslant d$.
- Φ a Kraus rank d quantum channel: If $\mathrm{rank}(\rho) = 1$, then $\mathrm{rank}(\Phi(\rho)) \leqslant d$.

Φ a Kraus rank d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.
$\lambda_1(\Phi),\ldots,\lambda_{n^2}(\Phi)$ eigenvalues of Φ, ordered s.t. $|\lambda_1(\Phi)| \geqslant \cdots \geqslant |\lambda_{n^2}(\Phi)|$.

Φ TP and unital $\Rightarrow \lambda_1(\Phi)=1$, with associated eigenstate the maximally mixed state I/n.
The *spectral expansion parameter* of Φ is $\lambda(\Phi) := |\lambda_2(\Phi)|$.

Observation: $\lambda(\Phi) = |\lambda_1(\Phi-\Pi)|$, where Π is the *maximally mixing channel* on $\mathcal{M}_n(\mathbf{C})$, i.e.
$\Pi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\, I/n \in \mathcal{M}_n(\mathbf{C})$.
$\longrightarrow \lambda(\Phi)$ is a distance measure between Φ and the maximally mixing channel.

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$\longrightarrow \Phi$ is both 'economical' and 'resembling' the maximally mixing channel.
For instance, the dynamics associated to Φ converges fast to equilibrium:
For any state ρ on $\mathbf{C}^n$, $\forall\, q \in \mathbf{N}$, $\|\Phi^q(\rho) - I/n\|_1 \leqslant \sqrt{n}\|\Phi^q(\rho) - I/n\|_2 \leqslant \sqrt{n}\lambda(\Phi)^q$.
exponential convergence, at rate $|\log \lambda(\Phi)|$ ↲

Plan

Constructions of optimal classical expanders

Fact: For any undirected d-regular graph G, $\lambda(G) \geqslant 2\sqrt{d-1}/d - o_n(1)$.
$\longrightarrow$ G is an optimal undirected expander (aka *Ramanujan graph*) if $\lambda(G) \leqslant 2\sqrt{d-1}/d$.

Question: Do Ramanujan graphs exist?

1. Explicit constructions of exactly Ramanujan graphs only for $d = p^m + 1$, p prime.
2. Random constructions of almost Ramanujan graphs for all d.
3. Existence of exactly Ramanujan graphs for all d.

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In fact, for large n, almost all undirected regular graphs are almost Ramanujan:

Theorem [Uniform random undirected regular graph (Friedman, Bordenave)]

Fix $d \in \mathbf{N}$. Let G be uniformly distributed on the set of undirected d-regular graphs on n vertices.
Then, for all $\varepsilon > 0$, $\mathbf{P}\left(\lambda(G) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon\right) = 1 - o_n(1)$.

Remarks: ↱ simpler model of random undirected regular graph

- First proven for the 'doubled edges permutation model': for $d \in \mathbf{N}$ even, pick $\sigma_1, \ldots, \sigma_{d/2} \in \mathcal{S}_n$ independent uniformly distributed and let G have edges $\{(k, \sigma_i(k)), (k, \sigma_i^{-1}(k))\}_{1 \leqslant k \leqslant n, 1 \leqslant i \leqslant d/2}$.
- Result remains true for G a random directed regular graph and for d growing with n, up to a constant multiplicative factor: $\mathbf{P}\left(\lambda(G) \leqslant C/\sqrt{d} + \varepsilon\right) = 1 - o_n(1)$.

Constructions of optimal self-adjoint quantum expanders

Fact: For any self-adjoint Kraus rank d unital quantum channel Φ, $\lambda(\Phi) \geqslant 2\sqrt{d-1}/d - o_n(1)$.
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First attempts at exhibiting explicit constructions (inspired by classical ones): not optimal.
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Question: How to sample a unital quantum channel randomly?

Idea: Pick $K_1,\dots,K_d \in \mathcal{M}_n(\mathbf{C})$ at random, under the constraints $\begin{cases} \sum_{i=1}^d K_i^* K_i = I \\ \sum_{i=1}^d K_i K_i^* = I \end{cases}$.

$\Phi : X \mapsto \sum_{i=1}^d K_i X K_i^*$ is a random Kraus rank (at most) d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.

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Theorem [Paired Haar unitaries as Kraus operators (Hastings)]

Fix $d \in \mathbf{N}$ even. Pick $U_1, \dots, U_{d/2} \in \mathcal{M}_n(\mathbf{C})$ independent Haar unitaries. Let $K_i = \frac{U_i}{\sqrt{d}}$, $1 \leqslant i \leqslant \frac{d}{2}$.
The self-adjoint unital quantum channel Φ associated to the K_i's, K_i^*'s satisfies:

$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon\right) = 1 - o_n(1).$$

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$$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon \right) = 1 - o_n(1).$$

Question: Can this result be extended to non-self-adjoint random models? to a wider variety of them? to a regime where d is not fixed but grows with n?

Constructions of optimal non-self-adjoint quantum expanders

Fact: For any Kraus rank d unital quantum channel Φ, $\lambda(\Phi) \geqslant 1/\sqrt{d} - o_n(1)$.

Theorem [Haar unitaries as Kraus operators (Pisier, Timhadjelt)**]**

Pick $U_1, \ldots, U_d \in \mathcal{M}_n(\mathbf{C})$ independent Haar unitaries. Let $K_i = \frac{U_i}{\sqrt{d}}$, $1 \leqslant i \leqslant d$.
The unital quantum channel Φ associated to the K_i's satisfies:
$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\lambda(\Phi) \leqslant \frac{1}{\sqrt{d}}(1+\varepsilon)\right) \geqslant 1 - e^{-c\varepsilon n^{1/12}}$.

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Question: Does this remain true for 'less random' unitary Kraus operators?

A probability measure μ on $U(n)$ is a k-*design* if $\mathbf{E}_{U\sim\mu}[U^{\otimes k}(\cdot)U^{*\otimes k}] = \mathbf{E}_{U\sim\mu_H}[U^{\otimes k}(\cdot)U^{*\otimes k}]$.
Haar measure on $U(n)$ ↵

Theorem [2-design unitaries as Kraus operators (Lancien)]

Pick $U_1, \dots, U_d \in \mathcal{M}_n(\mathbf{C})$ independent 2-design unitaries. Let $K_i = \frac{U_i}{\sqrt{d}}$, $1 \leqslant i \leqslant d$.
If $d \geqslant (\log n)^{8+\delta}$, the unital quantum channel Φ associated to the K_i's satisfies:
$$\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2}{\sqrt{d}}\left(1 + \frac{C}{(\log n)^{\delta/6}}\right)\right) \geqslant 1 - \frac{1}{n}.$$

Interest: Almost optimal quantum expander from random Kraus operators which are sampled according to a simple measure on the unitary group (uniform measure on explicit finite subset).

More random examples of optimal quantum expanders

Need: Extend the notion of expander to non-unital quantum channels (or else, too constraining).

Theorem [Gaussians as Kraus operators (Lancien/Pérez-García)]

Pick $G_1, \ldots, G_d \in \mathcal{M}_n(\mathbf{C})$ independent Gaussians. Let $\tilde{K}_i = \frac{G_i}{\sqrt{d}}$, $K_i = \tilde{K}_i \Sigma^{-1/2}$, $1 \leqslant i \leqslant d$.

i.i.d. Gaussian entries (mean 0, variance $1/n$) ↵ $\quad$ ↳ $= \sum_{j=1}^d \tilde{K}_j^* \tilde{K}_j$

The quantum channel Φ associated to the K_i's satisfies: $\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{C}{\sqrt{d}}\right) \geqslant 1 - e^{-cn}$.

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Question: Does this remain true for ‘less random’ Kraus operators with independent entries?

Theorem [Sparse random matrices as Kraus operators (Lancien/Youssef)]

↱ e.g. adjacency matrix of d-biregular graph G on n vertices s.t. $\lambda(G) \leqslant C/\sqrt{d}$

Fix $A \in \mathcal{M}_n(\mathbf{R})$ a bistochastic matrix s.t. $|\lambda_2(A)| \leqslant \frac{C}{\sqrt{d}}$. Let $W \in \mathcal{M}_n(\mathbf{C})$ be a random matrix with independent centered entries s.t. $\forall\, 1 \leqslant k, l \leqslant n$, $\mathbf{E}|W_{kl}|^2 = A_{kl}$, $(\mathbf{E}|W_{kl}|^p)^{1/p} \leqslant C' p^\beta A_{kl}$, $p \in \mathbf{N}$.

[$\beta = 0$: bounded, $\beta = 1/2$: sub-Gaussian, $\beta = 1$: sub-exponential] ↲

Pick $W_1,\ldots,W_d \in \mathcal{M}_n(\mathbf{C})$ independent copies of W. Let $\tilde{K}_i = \frac{W_i}{\sqrt{d}}$, $K_i = \tilde{K}_i \Sigma^{-1/2}$, $1 \leqslant i \leqslant d$.

If $d \geqslant (\log n)^8$, the quantum channel Φ associated to the K_i's satisfies: $\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{C''}{\sqrt{d}}\right) \geqslant 1 - \frac{1}{n}$.

Interest: Almost optimal quantum expanders from random Kraus operators which are sparse and whose entries have any distribution following the moments’ growth assumption.

Proof idea to show that $\mathbf{E}\lambda(\Phi) \leqslant C/\sqrt{d}$

Goal: In all cases, we want to upper bound $\mathbf{E}|\lambda_2(\Phi)| = \mathbf{E}|\lambda_1(\Phi - \Pi_{\rho^*})|$, where ρ_* is the fixed state of Φ and $\Pi_{\rho_*} : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\rho_* \in \mathcal{M}_n(\mathbf{C})$.
First step: Upper bound $\mathbf{E}|\lambda_1(\Phi - \mathbf{E}(\Phi))|$ (and then show that $\mathbf{E}(\Phi)$ is close to Π_{ρ^*}).

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- Observation 1: $|\lambda_1(\Psi)| \leqslant s_1(\Psi) = \|\Psi\|_\infty$.
- Observation 2: $\|\Psi\|_\infty = \|M_\Psi\|_\infty$, where for $\Psi : X \mapsto \sum_{i=1}^d K_i X L_i^*$, $M_\Psi = \sum_{i=1}^d K_i \otimes \bar{L}_i$.

[Identification $\Psi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C}) \equiv M_\Psi : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$ preserves the operator norm.]
$\longrightarrow$ We want to upper bound $\mathbf{E}\| \underbrace{M_\Phi - \mathbf{E}(M_\Phi)}_{=:X} \|_\infty$, where $M_\Phi = \sum_{i=1}^d K_i \otimes \bar{K}_i$ with the K_i's random.

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↱ Haar unitaries, Gaussians

- For concrete models, this can be done by a moments' method:

By Jensen's inequality, we have: $\forall\, p \in \mathbf{N}$, $\mathbf{E}\|X\|_\infty \leqslant \mathbf{E}\|X\|_p \leqslant (\mathbf{E}\,\mathrm{Tr}|X|^p)^{1/p}$.
The term on the r.h.s. can be estimated and provides a good upper bound for $p \simeq n^\gamma$.
↳ by Weingarten or Wick calculus

↱ 2-design unitaries, arbitrary sparse random matrices

- For more general cases, we use recent operator norm estimates for random matrices with dependencies and non-homogeneity (Brailovskaya/van Handel):

Setting $X = \sum_{i=1}^d Z_i$, with $Z_i := K_i \otimes \bar{K}_i - \mathbf{E}(K_i \otimes \bar{K}_i)$, $1 \leqslant i \leqslant d$, we have:

$$\mathbf{E}\|X\|_\infty \lesssim \|\mathbf{E}(XX^*)\|_\infty^{1/2} + \|\mathbf{E}(X^*X)\|_\infty^{1/2} + (\log n)^{3/2}\|\mathbf{Cov}(X)\|_\infty^{1/2} + (\log n)^6 \left(\mathbf{E}\max_{1\leqslant i\leqslant d}\|Z_i\|_\infty^2\right)^{1/2}.$$

Plan

Matrix product states (MPS) form a subset of *many-body quantum states*.

They are particularly useful because:

- They admit an *efficient description*: number of parameters that scales linearly rather than exponentially with the number of subsytems.
- They are *good approximations of several 'physically relevant' states*, such as ground states of gapped local Hamiltonians on 1D systems (Hastings, Landau/Vazirani/Vidick).
  - ↳ composed of terms which act non-trivially only on nearby sites
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with the distance separating the sites ←

between observables measured on distinct sites ←

Fact: Random (translation-invariant) MPS typically have correlations that decay exponentially fast, with a *small correlation length* (Lancien/Pérez-García).

Proof strategy: Observe that the correlation length is given by $1/|\log\lambda(\Phi)|$ for Φ a random quantum channel associated to the random MPS (its so-called *transfer operator*).

Some perspectives

- What about *explicit constructions* of optimal quantum expanders?
  Important for applications (cryptography, error correction, condensed matter physics, etc).

  Seminal constructions required a large amount of randomness.
  First step towards *derandomization*: Kraus operators sampled from an explicit finite subset of unitaries or as sparse matrices with independent ± 1 entries.

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- What about identifying the *full spectral distribution* of random quantum channels?
  ρ_* fixed state of Φ, $\Pi_{\rho_*} : X \mapsto \mathrm{Tr}(X)\rho_*$
  For a random Kraus rank d quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$, the eigenvalues of $\Phi - \Pi_{\rho_*}$ are typically inside a segment of half-length $2\sqrt{d-1}/d$ or a disc of radius $1/\sqrt{d}$ for large n.
  ↳ self-adjoint case ↳ non-self-adjoint case
  But what is their exact distribution inside this segment or disc?
  Answer: As $n, d \to \infty$, the spectral distribution of $\sqrt{d}(\Phi - \Pi_{\rho_*})$ is proven to be semi-circular in the self-adjoint case (Lancien/Oliveira Santos/Youssef) and conjectured to be circular in the non-self-adjoint case (Bruzda/Cappellini/Sommers/Życzkowski, Aubrun/Nechita).

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  First step towards *derandomization*: Kraus operators sampled from an explicit finite subset of unitaries or as sparse matrices with independent ± 1 entries.

- What about identifying the *full spectral distribution* of random quantum channels?

  ρ_* fixed state of Φ, $\Pi_{\rho_*} : X \mapsto \mathrm{Tr}(X)\rho_*$

  For a random Kraus rank d quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$, the eigenvalues of $\Phi - \Pi_{\rho_*}$ are typically inside a segment of half-length $2\sqrt{d-1}/d$ or a disc of radius $1/\sqrt{d}$ for large n.

  ↳ self-adjoint case ↳ non-self-adjoint case

  But what is their exact distribution inside this segment or disc?
  Answer: As $n, d \to \infty$, the spectral distribution of $\sqrt{d}(\Phi - \Pi_{\rho_*})$ is proven to be semi-circular in the self-adjoint case (Lancien/Oliveira Santos/Youssef) and conjectured to be circular in the non-self-adjoint case (Bruzda/Cappellini/Sommers/Życzkowski, Aubrun/Nechita).

- Do the results about the typical spectral gap of random quantum channels remain true when we impose *extra symmetries* on the model?

Some perspectives

- What about *explicit constructions* of optimal quantum expanders?
  Important for applications (cryptography, error correction, condensed matter physics, etc).

  Seminal constructions required a large amount of randomness.
  First step towards *derandomization*: Kraus operators sampled from an explicit finite subset of unitaries or as sparse matrices with independent ± 1 entries.

- What about identifying the *full spectral distribution* of random quantum channels?

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- Do the results about the typical spectral gap of random quantum channels remain true when we impose *extra symmetries* on the model?

- What about looking at other, related, notions of expansions, such as *geometric* ones (Bannink/Briët/Labib/Maassen) or *linear-algebraic* ones (Li/Qiao/Wigderson/Wigderson/Zhang)?

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