Random quantum correlations are generically non-classical

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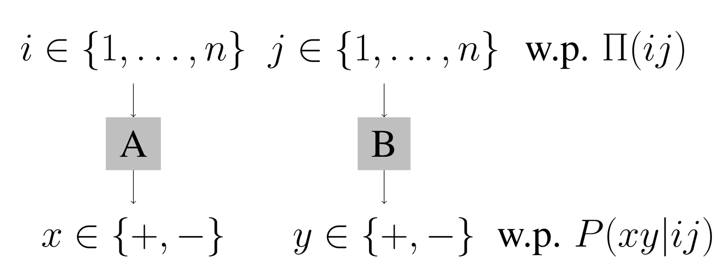
1 Classical and quantum correlation matrices

2-player 2-outcome non-local game



A & B gain $V(ijxy) = \begin{cases} +V(ij) \text{ if } x = y \\ -V(ij) \text{ if } x \neq y \end{cases}$

Given a strategy (i.e. conditional p.d.) P for A & B, the associated correlation τ is the $n \times n$ matrix s.t., for each $i,j \in \{1,\ldots,n\}$,

$$\tau_{ij} = P(++|ij) + P(--|ij) - P(+-|ij) - P(-+|ij)$$

Goal of A & B: $\max \left\{ \sum_{i,j=1}^{n} \Pi(ij)V(ij)\tau_{ij},\ P \text{ allowed} \right\}$.

Allowed strategies and associated correlations

- **Classical strategy:** For each $x,y \in \{+,-\}$ and $i,j \in \{1,\ldots,n\}$, $P(xy|ij) = \sum_\lambda q_\lambda A(x|i\lambda)B(y|j\lambda)$, with $\{q_\lambda\}_\lambda, \{A(+|i\lambda), A(-|i\lambda)\}, \{B(+|j\lambda), B(-|j\lambda)\}$ p.d.'s.
- **Quantum strategy:** For each $x,y \in \{+,-\}$ and $i,j \in \{1,\ldots,n\}$, $P(xy|ij) = \mathrm{Tr}(A_i^x \otimes B_j^y \varrho)$, with ϱ state on $\mathcal{H}_A \otimes \mathcal{H}_B$, $(A_i^+, A_i^-), (B_j^+, B_j^-)$ POVMs on $\mathcal{H}_A, \mathcal{H}_B$.
- **Classical correlation:** $\tau \in \mathcal{C} := \left\{ \left(\mathbf{E}[X_iY_j]\right)_{1\leqslant i,j\leqslant n},\ |X_i|, |Y_j| \leqslant 1 \text{ a.s.} \right\}$.
- **Quantum correlation:** $\tau \in \mathcal{Q} := \left\{ \left(\mathrm{Tr}[X_i \otimes Y_j\, \varrho]\right)_{1\leqslant i,j\leqslant n},\ \begin{cases} \|X_i\|_\infty, \|Y_j\|_\infty \leqslant 1 \\ X_i^* = X_i,\ Y_j^* = Y_j \end{cases},\ \varrho \text{ state} \right\}$.

Proposition 1.1 (Characterization of $\mathcal{C}$ & $\mathcal{Q}$).

$$\mathcal{C} = \mathrm{conv}\left\{(\alpha_i\beta_j)_{1\leqslant i,j\leqslant n},\ \alpha_i,\beta_j = \pm 1\right\} \ \&\ \mathcal{Q} = \mathrm{conv}\left\{(\langle u_i|v_j\rangle)_{1\leqslant i,j\leqslant n},\ u_i,v_j \in S_{\mathbf{R}^m}\right\}$$

2 Correlation matrices and tensor norms

Definition/Proposition 2.1 (The dual norms $\ell_1^n \otimes_\epsilon \ell_1^n$ & $\ell_\infty^n \otimes_\pi \ell_\infty^n$).

$$\|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n} := \sup\left\{ \sum_{i,j=1}^{n} M_{ij}\alpha_i\beta_j,\ \alpha_i,\beta_j = \pm 1 \right\} \ \&\ \|\tau\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} := \inf\left\{ \sum_{k=1}^{N} \|x_k\|_\infty\|y_k\|_\infty,\ \tau = \sum_{k=1}^{N} x_k \otimes y_k \right\}$$

$$\tau \in \mathcal{C} \ \Leftrightarrow\ \forall\, M \text{ s.t. } \|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n} \leqslant 1,\ \mathrm{Tr}(\tau M^t) \leqslant 1 \ \Leftrightarrow\ \|\tau\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} \leqslant 1$$

Definition/Proposition 2.2 (The dual norms γ_2^* & γ_2).

$$\gamma_2^*(M) := \sup\left\{ \sum_{i,j=1}^{n} M_{ij}\langle u_i|v_j\rangle,\ u_i,v_j \in S_{\mathbf{R}^m} \right\} \ \&\ \gamma_2(\tau) := \inf\left\{ \max_{1\leqslant i\leqslant n}\|R_i(X)\|_2 \max_{1\leqslant j\leqslant n}\|C_j(Y)\|_2,\ \tau = XY \right\}$$

$$\tau \in \mathcal{Q} \ \Leftrightarrow\ \forall\, M \text{ s.t. } \gamma_2^*(M) \leqslant 1,\ \mathrm{Tr}(\tau M^t) \leqslant 1 \ \Leftrightarrow\ \gamma_2(\tau) \leqslant 1$$

Known: By Grothendieck's inequality [5], there exists $1.67 < K_G < 1.79$ s.t., for any $n \times n$ matrix T,

$$\gamma_2(T) \leqslant \|T\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} \leqslant K_G\gamma_2(T).$$

$\rightarrow$ No unbounded ratio between the "classical" and "quantum" norms of T.

Question: What typically happens for T picked at random? In particular, can the dominating constant in the first inequality be improved from 1 to a value strictly bigger than 1?

3 Main result

Theorem 3.1. *Let T be an $n \times n$ random matrix satisfying the two assumptions: (1) its distribution is bi-orthogonally invariant, and (2) w.h.p. $\|T\|_\infty \leqslant (r + o(1))\frac{\|T\|_1}{n}$. Then w.h.p. as $n \to +\infty$,*

$$\|T\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} \geqslant \left(\sqrt{\frac{16}{15}} - o(1)\right)\gamma_2(T) > \gamma_2(T).$$

Consequence: The random correlation $\tau = \frac{T}{\gamma_2(T)}$ is quantum (by construction) but w.h.p. non-classical.

Examples of applications:

- Let G be an $n \times n$ Gaussian matrix.

$\tau = \frac{G}{\gamma_2(G)}$ is uniformly distributed on the border of $\mathcal{Q}$ but w.h.p. not in $\mathcal{C}$.
$\rightarrow$ The borders of $\mathcal{C}$ and $\mathcal{Q}$ do not coincide in typical directions.

- Let $u_1,\ldots,u_n,v_1,\ldots,v_n$ be independent and uniformly distributed unit vectors in $\mathbf{R}^m$.

$\tau = (\langle u_i|v_j\rangle)_{1\leqslant i,j\leqslant n}$ is in $\mathcal{Q}$ but w.h.p. not in $\mathcal{C}$ if $m/n < 0.13$.
$\rightarrow$ Bridging the gap between this result and the opposite one from [3], stating that τ is w.h.p. in $\mathcal{C}$ if $m/n > 2$?

Two main technical lemmas needed:

- SVD of a bi-orthogonally invariant random matrix T [3]:

$T \sim U\Sigma V^t$ with U,V,Σ independent, U,V uniformly distributed orthogonal matrices, Σ diagonal positive semidefinite matrix.

- Levy's lemma for an L-Lipschitz function $f : S_{\mathbf{R}^n} \to \mathbf{R}$ with median M_f (w.r.t. the uniform measure) [2]:

$\forall\, 0 < \theta < \pi/2,\ \mathbf{P}\left(f \gtrless M_f \pm (\cos\theta)L\right) \leqslant \frac{1}{2}(\sin\theta)^{n-1} \leqslant \frac{1}{2}e^{-(n-1)(\cos\theta)^2/2}$.

4 Upper bounding the quantum norm of a random matrix

Proposition 4.1. *Let T be an $n \times n$ random matrix satisfying (1) and (2). Then w.h.p. as $n \to +\infty$,*

$$\gamma_2(T) \leqslant \left(1 + o(1)\right)\frac{\|T\|_1}{n}.$$

Main steps in the proof:

- SVD of T: $T = XY$ with $X = U\sqrt{\Sigma}, Y = \sqrt{\Sigma}V^t$.
- Levy's lemma: $\forall\, 1 \leqslant i,j \leqslant n$, $\begin{cases} \mathbf{P}\left(\|R_i(X)\|_2^2 > (1+\epsilon)\frac{\mathrm{Tr}\,\Sigma}{n}\right) \leqslant e^{-cn\epsilon^2/r^2} \\ \mathbf{P}\left(\|C_j(Y)\|_2^2 > (1+\epsilon)\frac{\mathrm{Tr}\,\Sigma}{n}\right) \leqslant e^{-cn\epsilon^2/r^2} \end{cases}$.

So by the union bound (on $2n$ events):

$$\mathbf{P}\left(\gamma_2(T) \leqslant (1+\epsilon)\frac{\mathrm{Tr}\,\Sigma}{n}\right) \geqslant \mathbf{P}\left(\forall\, 1 \leqslant i,j \leqslant n,\ \|R_i(X)\|_2\|C_j(Y)\|_2 \leqslant (1+\epsilon)\frac{\mathrm{Tr}\,\Sigma}{n}\right) \geqslant 1 - 2n\,e^{-cn\epsilon^2/r^2}.$$

Remark: This result is optimal.
Indeed, by duality: $\gamma_2(T) = \sup_M \frac{\mathrm{Tr}(TM^t)}{\gamma_2^*(M)} \geqslant \frac{\mathrm{Tr}\,\Sigma}{\gamma_2^*(UV^t)}$, where $T = U\Sigma V^t$ (taking $M = UV^t$).
And for any $n \times n$ orthogonal matrix O, $\gamma_2^*(O) = n$.

5 Lower bounding the classical norm of a random matrix

Proposition 5.1. *Let T be an $n \times n$ random matrix satisfying (1). Then w.h.p. as $n \to +\infty$,*

$$\|T\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} \geqslant \left(\sqrt{\frac{16}{15}} - o(1)\right)\frac{\|T\|_1}{n}.$$

Main steps in the proof:

- Duality: $\|T\|_{\ell_\infty^n\otimes_\pi\ell_\infty^n} = \sup_M \frac{\mathrm{Tr}(TM^t)}{\|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n}} \geqslant \frac{\mathrm{Tr}\,\Sigma}{\|UV^t\|_{\ell_1^n\otimes_\epsilon\ell_1^n}}$, where $T = U\Sigma V^t$ (taking $M = UV^t$).
- Levy's lemma: $\forall\, \alpha,\beta \in \{\pm 1\}^n$, $\mathbf{P}\left(\sum_{i,j=1}^{n}(UV^t)_{ij}\alpha_i\beta_j > (\cos\theta)n\right) \leqslant \frac{1}{2}(\sin\theta)^{n-1}$.

So by the union bound (on 4^n events):

$$\mathbf{P}\left(\|UV^t\|_{\ell_1^n\otimes_\epsilon\ell_1^n} \leqslant (\cos\theta)n\right) = \mathbf{P}\left(\forall\, \alpha,\beta \in \{\pm 1\}^n,\ \sum_{i,j=1}^{n}(UV^t)_{ij}\alpha_i\beta_j \leqslant (\cos\theta)n\right) \geqslant 1 - 4^n\frac{1}{2}(\sin\theta)^{n-1}.$$

And $4\sin\theta < 1 \ \Leftrightarrow\ \cos\theta > \sqrt{15/16}$.

Remark: This result is potentially non-optimal for two reasons.

- Is there a better choice than $M = UV^t$ as Bell functional?
- What is the exact asymptotics of $\mathbf{E}\|O\|_{\ell_1^n\otimes_\epsilon\ell_1^n}$ for O an $n \times n$ uniformly distributed orthogonal matrix?

We only know that $\left(\sqrt{\frac{2}{\pi}} - o(1)\right)n \leqslant \mathbf{E}\|O\|_{\ell_1^n\otimes_\epsilon\ell_1^n} \leqslant \left(\sqrt{\frac{15}{16}} + o(1)\right)n$.

6 Concluding remarks and perspectives

- Given T a random matrix satisfying (1) and (2), we can exhibit a Bell functional M generically witnessing the generic non-classicality of the quantum correlation $\tau = \frac{T}{\gamma_2(T)}$, namely $M = UV^t$ where $T = U\Sigma V^t$ is the SVD of T.
- **Dual problem:** Given a random Bell functional M, is its quantum value (i.e. $\gamma_2^*(M)$) w.h.p. strictly bigger than its classical value (i.e. $\|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n}$)?

Answer from [1]: If M is an $n \times n$ Gaussian or Bernoulli matrix, then w.h.p. as $n \to +\infty$,

$$\gamma_2^*(M) \geqslant \left(\frac{1}{\sqrt{\ln 2}} - o(1)\right)\|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n} > \|M\|_{\ell_1^n\otimes_\epsilon\ell_1^n}.$$

- **Weaker corollaries:** Separations of $\mathcal{Q}^*$ vs $\mathcal{C}^*$ and $\mathcal{Q}$ vs $\mathcal{C}$ in terms of mean width w. Namely,

$$w(\mathcal{Q}^*) < w(\mathcal{C}^*) \text{ and } w(\mathcal{Q}) > w(\mathcal{C}).$$

Definition: Given $\mathcal{K}$ a set of $n \times n$ matrices, $w(\mathcal{K}) := \mathbf{E}\sup_{X\in\mathcal{K}} \mathrm{Tr}(GX^t)$, for G an $n \times n$ Gaussian matrix.

- For much more around this topic, see [4].

References

[1] **A. Ambainis, A. Bačkurs, K. Balodis, D. Kravčenko, R. Ozols, J. Smotrovs, M. Virza**, "Quantum strategies are better than classical in almost any XOR games".

[2] **G. Aubrun, S.J. Szarek**, *Alice and Bob meet Banach*.

[3] **C.E. González-Guillén, C.H. Jiménez, C. Palazuelos, I. Villanueva**, "Sampling quantum nonlocal correlations with high probability".

[4] **C. Palazuelos, T. Vidick**, "Survey on non-local games and operator space theory".

[5] **G. Pisier**, "Grothendieck's theorem, past and present".