Quantum expanders – Random constructions & Applications

Based on the works:

• arXiv:1906.11682 (with David Pérez-Garcìa)
• arXiv:2302.07772 (with Pierre Youssef)
• arXiv:2409.17971

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Plan

Classical expanders

G a (directed or undirected) d-biregular graph on n vertices.

↳ d incoming and d outgoing edges at each vertex

A its (normalized) adjacency matrix, i.e. the $n \times n$ matrix s.t. $A_{kl} = e(l \to k)/d$ for all $1 \leqslant k, l \leqslant n$.

number of edges from vertex l to vertex k ↲

$\lambda_1(A), \ldots, \lambda_n(A)$ eigenvalues of A, ordered s.t. $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$.

G biregular $\Rightarrow$ A bistochastic $\Rightarrow$ $\lambda_1(A) = 1$, with associated eigenvector the uniform probability u.
The *spectral expansion parameter* of G is $\lambda(G) := |\lambda_2(A)|$. $(1/n, \ldots, 1/n) =$ ↲

Observation: $\lambda(G) = |\lambda_1(A - J)|$, where J is the adjacency matrix of the *complete graph* on n vertices, i.e. the $n \times n$ matrix whose entries are all equal to $1/n$.
$\longrightarrow$ $\lambda(G)$ is a distance measure between G and the complete graph.

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Definition [Classical expander (informal)]

A d-biregular graph G on n vertices is an *expander* if it is sparse (i.e. $d \ll n$) and spectrally expanding (i.e. $\lambda(G) \ll 1$).

$\longrightarrow$ G is both 'economical' and 'resembling' the complete graph.
For instance, a random walk supported on G converges fast to equilibrium:
For any probability p on $\{1, \ldots, n\}$, $\forall\, q \in \mathbf{N}$, $\|A^q p - u\|_1 \leqslant \sqrt{n}\lambda(G)^q$.

↳ exponential convergence, at rate $|\log \lambda(G)|$

Quantum analogue of the transition matrix associated to a biregular graph

Classical - Quantum correspondence:

- $p \in \mathbf{R}^n$ probability vector $\leftrightsquigarrow$ $\rho \in \mathcal{M}_n(\mathbf{C})$ *quantum state*.
  ↳ self-adjoint positive semidefinite trace 1 matrix
- $A : \mathbf{R}^n \to \mathbf{R}^n$ transition matrix $\leftrightsquigarrow$ $\Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$ *quantum channel*.
  ↳ completely positive (CP) trace-preserving (TP) linear map
- G biregular: A leaves u invariant $\leftrightsquigarrow$ Φ unital: Φ leaves I/n invariant.

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Question: What is the analogue of the degree in the quantum setting?

Answer: The Kraus rank.
Given a CP map Φ on $\mathcal{M}_n(\mathbf{C})$, a *Kraus representation* of Φ is of the form:

$$\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^{d} K_i X K_i^* \in \mathcal{M}_n(\mathbf{C}), \text{ where } K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C}). \quad (\star)$$

↳ Kraus operators of Φ

The minimal d s.t. Φ can be written as $(\star)$ is the *Kraus rank* of Φ (it is always at most n^2).
[Note: Φ is TP iff $\sum_{i=1}^d K_i^* K_i = I$. Φ is unital iff $\sum_{i=1}^d K_i K_i^* = I$.]

Indeed, the degree and the Kraus rank both quantify the 1-iteration spreading:

- G a degree d graph: If $|\mathrm{supp}(p)| = 1$, then $|\mathrm{supp}(Ap)| \leqslant d$.
- Φ a Kraus rank d quantum channel: If $\mathrm{rank}(\rho) = 1$, then $\mathrm{rank}(\Phi(\rho)) \leqslant d$.

Quantum expanders

Φ a Kraus rank d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.
$\lambda_1(\Phi),\ldots,\lambda_{n^2}(\Phi)$ eigenvalues of Φ, ordered s.t. $|\lambda_1(\Phi)|\geqslant\cdots\geqslant|\lambda_{n^2}(\Phi)|$.

Φ TP and unital $\Rightarrow \lambda_1(\Phi)=1$, with associated eigenstate the maximally mixed state I/n.
The *spectral expansion parameter* of Φ is $\lambda(\Phi):=|\lambda_2(\Phi)|$.

Observation: $\lambda(\Phi)=|\lambda_1(\Phi-\Pi)|$, where Π is the *maximally mixing channel* on $\mathcal{M}_n(\mathbf{C})$, i.e.
$\Pi : X\in\mathcal{M}_n(\mathbf{C})\mapsto \mathrm{Tr}(X)\, I/n\in\mathcal{M}_n(\mathbf{C})$.
$\longrightarrow \lambda(\Phi)$ is a distance measure between Φ and the maximally mixing channel.

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For instance, the dynamics associated to Φ converges fast to equilibrium:
For any state ρ on $\mathbf{C}^n$, $\forall\, q \in \mathbf{N}$, $\|\Phi^q(\rho) - I/n\|_1 \leqslant \sqrt{n}\lambda(\Phi)^q$.
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Plan

1. Classical and quantum expanders: definitions and motivations
2. Random constructions of expanders
3. Applications and perspectives

Optimal expanders: definition and constructions in the classical case

(or $2\sqrt{d-1}/d$ for G undirected)

Fact: For any d-biregular graph G on n vertices, $\lambda(G) \geqslant 1/\sqrt{d} - o_n(1)$.
$\longrightarrow$ G is an optimal classical expander (aka *Ramanujan graph*) if $\lambda(G) \leqslant 1/\sqrt{d}$.

Question: Do Ramanujan graphs exist?

1. Explicit constructions of exactly Ramanujan graphs only for $d = p^m + 1$, p prime.
2. Random constructions of almost Ramanujan graphs for all d. More precisely: for large n, almost all d-biregular graphs are almost Ramanujan (Friedman, Bordenave).
3. Existence of exactly Ramanujan graphs for all d.

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(or $2\sqrt{d-1}/d$ for Φ self-adjoint)

Fact: For any Kraus rank d unital quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$, $\lambda(\Phi) \geqslant 1/\sqrt{d} - o_n(1)$.
$\longrightarrow$ Φ is an optimal quantum expander if $\lambda(\Phi) \leqslant 1/\sqrt{d}$.

Question: Do optimal quantum expanders exist?
First attempts at exhibiting explicit constructions (inspired by classical ones): not optimal.
$\longrightarrow$ What about random constructions?

Strategy: Pick $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$ at random, under the constraints $\begin{cases} \sum_{i=1}^d K_i^* K_i = I \\ \sum_{i=1}^d K_i K_i^* = I \end{cases}$.

$\Phi : X \mapsto \sum_{i=1}^d K_i X K_i^*$ is a random Kraus rank (at most) d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.

Constructions of optimal expanders in the quantum case

Theorem [Haar unitaries as Kraus operators (Hastings, Pisier, Timhadjelt)]

Pick $U_1, \ldots, U_d \in \mathcal{M}_n(\mathbf{C})$ independent Haar unitaries. Let $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d$.
The unital quantum channel Φ associated to the K_i's satisfies:
$\forall\, \varepsilon > 0,\ \mathbf{P}\left(\lambda(\Phi) \leqslant \frac{1}{\sqrt{d}}(1+\varepsilon)\right) \geqslant 1 - e^{-c\varepsilon n^{1/12}}$, where $c > 0$ is an absolute constant.

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Question: Does this remain true for 'less random' unitary Kraus operators?

A measure μ on $U(n)$ is a k-*design* if $\mathbf{E}_{U\sim\mu}[U^{\otimes k}(\cdot)U^{*\otimes k}] = \mathbf{E}_{U\sim\mu_H}[U^{\otimes k}(\cdot)U^{*\otimes k}]$.
↳ Haar measure on $U(n)$

Theorem [2-design unitaries as Kraus operators (Lancien)]

Pick $U_1,\ldots,U_d \in \mathcal{M}_n(\mathbf{C})$ independent 2-design unitaries. Let $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d$.
If $d \geqslant (\log n)^{8+\delta}$ for some $\delta > 0$, the unital quantum channel Φ associated to the K_i's satisfies:
$\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2}{\sqrt{d}}\left(1 + \frac{C}{(\log n)^{\delta/6}}\right)\right) \geqslant 1 - \frac{1}{n}$, where $C < \infty$ is an absolute constant.

Interest: Nearly optimal quantum expander from random Kraus operators which are sampled according to a simple measure on the unitary group (uniform measure on explicit finite subset).

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Interest: Nearly optimal quantum expander from random Kraus operators which are sampled according to a simple measure on the unitary group (uniform measure on explicit finite subset).

Remark: Similar results for Kraus operators sampled as (renormalized) Gaussian matrices (Lancien/Pérez-García) or sparse matrices with arbitrary independent entries (Lancien/Youssef).

Proof idea to show that $\mathbf{E}\lambda(\Phi) \leqslant 2/\sqrt{d}$

$\ulcorner\!\!\rightarrow = \mathbf{E}(\Phi)$

Goal: Upper bound $\mathbf{E}|\lambda_2(\Phi)| = \mathbf{E}|\lambda_1(\Phi - \Pi)|$, where $\Pi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\, I/n \in \mathcal{M}_n(\mathbf{C})$.
(Next: Concentration of measure to show that $\lambda(\Phi)$ is typically close to $\mathbf{E}\lambda(\Phi)$...)

Observations:

- $|\lambda_1(\Psi)| \leqslant s_1(\Psi) = \|\Psi\|_\infty$.
- $\|\Psi\|_\infty = \|M_\Psi\|_\infty$, where for $\Psi : X \mapsto \sum_{i=1}^d K_i X L_i^*$, $M_\Psi = \sum_{i=1}^d K_i \otimes \bar{L}_i$.

[Identification $\Psi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C}) \equiv M_\Psi : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$ preserves the operator norm.]

$\longrightarrow$ Upper bound $\mathbf{E}\| \underbrace{M_\Phi - \mathbf{E}(M_\Phi)}_{=:X} \|_\infty$, where $M_\Phi = \sum_{i=1}^d K_i \otimes \bar{K}_i$.

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Implementation:

- For Haar unitaries, this can be done by a *moments' method*:

By Jensen's inequality, we have: $\forall\, p \in \mathbf{N},\ \mathbf{E}\|X\|_\infty \leqslant \mathbf{E}\|X\|_p \leqslant (\mathbf{E}\,\mathrm{Tr}|X|^p)^{1/p}$.

$\longrightarrow$ Estimate the r.h.s. by *Weingarten calculus* and choose $p = p_{n,d}$ that minimizes it.

- For 2-design unitaries, we use recent *operator norm estimates for random matrices with dependencies and non-homogeneity* (Brailovskaya/van Handel):

Setting $X = \sum_{i=1}^d Z_i$, with $Z_i := K_i \otimes \bar{K}_i - \mathbf{E}(K_i \otimes \bar{K}_i)$, $1 \leqslant i \leqslant d$, we have:

$$\mathbf{E}\|X\|_\infty \leqslant \|\mathbf{E}(XX^*)\|_\infty^{1/2} + \|\mathbf{E}(X^*X)\|_\infty^{1/2} + C(\log n)^6 \left(\|\mathbf{Cov}(X)\|_\infty^{1/2} + \left(\mathbf{E} \max_{1 \leqslant i \leqslant d} \|Z_i\|_\infty^2 \right)^{1/2} \right).$$

$\longrightarrow$ Estimate all parameters appearing on the r.h.s.

Plan

Implications for typical decay of correlations in many-body quantum systems

Matrix product states (MPS) form a subset of *many-body quantum states*.

They are particularly useful because:

- They admit an *efficient description*: number of parameters that scales linearly rather than exponentially with the number of subsytems.
- They are *good approximations of several 'physically relevant' states*, such as ground states of gapped local Hamiltonians on 1D systems (Hastings, Landau/Vazirani/Vidick).
  - ↳ composed of terms which act non-trivially only on nearby sites
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  ↳ spectral gap lower bounded by a constant independent of the number of subsystems

with the distance separating the sites ←

between observables measured on distinct sites ←

Fact: Random (translation-invariant) MPS typically have correlations that decay exponentially fast, with a *small correlation length* (Lancien/Pérez-García).

Proof strategy: Observe that the correlation length is upper bounded by $1/|\log \lambda(\Phi)|$ for Φ a random quantum channel associated to the random MPS (its so-called *transfer operator*).

Some perspectives

- What about *explicit constructions* of optimal quantum expanders?
  Important for applications (cryptography, error correction, condensed matter physics, etc.)

  Seminal constructions required a large amount of randomness.
  First step towards *derandomization*: Kraus operators sampled from the Clifford group or as sparse matrices with independent ± 1 entries.

  Generalization: Nearly optimal *k-copy quantum expander* with Kraus operators sampled from a unitary $2k$-design (Harrow/Hastings, Fukuda, Lancien).
  ↳ efficiently generable: e.g. random circuit of depth $\mathrm{poly}(\log n, k)$ (Haferkamp/Huang/Schuster)

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- What about identifying the *full spectral distribution* of random quantum channels?
  For large n, the eigenvalues of $\sqrt{d}(\Phi - \Pi)$ are typically inside the unit disc.
  But how are they distributed? ↳ random Kraus rank d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$

  Full answer in the self-adjoint case (Lancien/Oliveira Santos/Youssef).
  Partial conjectures in the non-self-adjoint case (Bruzda/Cappellini/Sommers/Życzkowski).

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- Do the results about the typical spectral gap of random quantum channels remain true when we impose *extra symmetries* on the model?

- What about looking at other, related, notions of expansions, such as more *geometric* ones (Bannink/Briët/Labib/Maassen) or *linear-algebraic* ones (Li/Qiao/Wigderson/Wigderson/Zhang)?

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