Exercises – October 21st 2025

Exercise 1. *We recall that the partial trace* $\mathrm{Tr}_{\mathcal{H}_2} : B(\mathcal{H}_1 \otimes \mathcal{H}_2) \to B(\mathcal{H}_1)$ *is the linear map whose action on product operators is* $\mathrm{Tr}_{\mathcal{H}_2}(X_1 \otimes X_2) = \mathrm{Tr}(X_2)X_1$ *and is extended to* $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *by linearity.*

1. *Show that* $\mathrm{Tr}_{\mathcal{H}_2}$ *satisfies*

$$\forall\ X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2), Y \in B(\mathcal{H}_1),\ \mathrm{Tr}(X(Y \otimes \mathrm{I})) = \mathrm{Tr}\left(\mathrm{Tr}_{\mathcal{H}_2}(X)Y\right).$$

2. *Use the above observation to deduce that*

$$\forall\ X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2),\ X \geqslant 0 \implies \mathrm{Tr}_{\mathcal{H}_2}(X) \geqslant 0.$$

Exercise 2. *Given a unit vector* $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$*, we define its distance to the set of unit product vectors as*

$$d(\varphi) = \min\left\{\|\varphi - \chi_1 \otimes \chi_2\| : \chi_1, \chi_2 \in \mathbb{C}^d,\ \|\chi_1\|, \|\chi_2\| = 1\right\}.$$

What is the maximal value that $d(\varphi)$ *can take? Show that it is attained iff* φ *is maximally entangled.*

Exercise 3. *Denote by* $\{e_1, \ldots, e_d\}$ *the canonical orthonormal basis of* $\mathbb{C}^d$*, and let* $\psi = \frac{1}{\sqrt{d}}\sum_{i=1}^d e_i \otimes e_i$ *be a maximally entangled unit vector in* $\mathbb{C}^d \otimes \mathbb{C}^d$*. Show that, for all* $X, Y \in B(\mathbb{C}^d)$*, we have*

$$\mathrm{Tr}\left(|\psi\rangle\langle\psi|(X \otimes Y)\right) = \frac{1}{d}\,\mathrm{Tr}\left(XY^T\right),$$

where Y^T *denotes the transposition of* Y*, with respect to the basis* $\{e_1, \ldots, e_d\}$*.*

Exercise 4. *[PPT criterion for entanglement]*
We recall that, given an orthonormal basis $\{e_1, \ldots, e_{d_1}\}$ *of* $\mathcal{H}_1$*, the transposition with respect to this basis is the linear map* $T : B(\mathcal{H}_1) \to B(\mathcal{H}_1)$ *that acts as*

$$T\left(\sum_{i,j=1}^{d_1} X_{ij}|e_i\rangle\langle e_j|\right) = \sum_{i,j=1}^{d_1} X_{ij}|e_j\rangle\langle e_i|.$$

We then define the partial transposition as the linear map $\Gamma : B(\mathcal{H}_1 \otimes \mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *that acts as the transposition on* $B(\mathcal{H}_1)$ *and the identity on* $B(\mathcal{H}_2)$*. Concretely, it acts as*

$$\Gamma\left(\sum_{i_1,j_1=1}^{d_1}\sum_{i_2,j_2=1}^{d_2} X_{i_1j_1i_2j_2}|e_{i_1}\rangle\langle e_{j_1}| \otimes |f_{i_2}\rangle\langle f_{j_2}|\right) = \sum_{i_1,j_1=1}^{d_1}\sum_{i_2,j_2=1}^{d_2} X_{i_1j_1i_2j_2}|e_{j_1}\rangle\langle e_{i_1}| \otimes |f_{i_2}\rangle\langle f_{j_2}|.$$

[Note: The (partial) transposition depends on the choice of basis, but the spectrum of the (partial) transposition of a matrix does not. One crucial difference between transposition and partial transposition, though, is that transposition preserves the spectrum while partial transposition generally does not.]

A state ρ *on* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *is called positive under partial transposition (PPT) if* $\Gamma(\rho) \geqslant 0$*.*

1. *Given a state* ρ *on* $\mathcal{H}_1 \otimes \mathcal{H}_2$*, show that if it is separable, then it is PPT.*
   [This fact is usually used in its contrapositive form, as a so-called entanglement criterion, namely: if ρ *is not PPT, then it is guaranteed to be entangled.]*
2. *Given a pure state* $\rho = |\varphi\rangle\langle\varphi|$ *on* $\mathcal{H}_1 \otimes \mathcal{H}_2$*, show that it is separable iff it is PPT.*

Exercise 5. *[Flip operator]*
We define the flip operator F on $\mathbb{C}^d\otimes\mathbb{C}^d$ as the operator whose action on product vectors is $F(\varphi_1\otimes\varphi_2)=\varphi_2\otimes\varphi_1$ and is extended to $\mathbb{C}^d\otimes\mathbb{C}^d$ by linearity.

1. *Given an orthonormal basis $\{e_1,\ldots,e_d\}$ of $\mathbb{C}^d$, write the expression of F in the orthonormal product basis $\{e_i\otimes e_j,\ 1\leqslant i,j\leqslant d\}$ of $\mathbb{C}^d\otimes\mathbb{C}^d$.*
2. *What are the eigenvalues and eigenvectors of F?*
3. *Show that F satisfies, for all $X,Y\in B(\mathbb{C}^d)$,*
$$\mathrm{Tr}\left(F(X\otimes Y)\right)=\mathrm{Tr}\left(XY\right).$$

Exercise 6. *Denote by $\{e_1,\ldots,e_d\}$ the canonical orthonormal basis of $\mathbb{C}^d$, and let $\psi=\frac{1}{\sqrt{d}}\sum_{i=1}^d e_i\otimes e_i$ be a maximally entangled unit vector in $\mathbb{C}^d\otimes\mathbb{C}^d$. Show that*
$$\Gamma(|\psi\rangle\langle\psi|)=\frac{1}{d}F,$$
where Γ is the partial transposition (as defined in Exercise 4) and F is the flip operator (as defined in Exercise 5).

Exercise 7. *[Isotropic states]*
Isotropic states on $\mathbb{C}^d\otimes\mathbb{C}^d$ are states which are convex combinations of the maximally mixed state I/d^2 and the maximally entangled state $|\psi\rangle\langle\psi|$. They have the form
$$\rho_\alpha=\alpha|\psi\rangle\langle\psi|+(1-\alpha)\frac{\mathrm{I}}{d^2},\quad \alpha\in[0,1].$$
Compute $\Gamma(\rho_\alpha)$ and show that ρ_α is PPT iff $\alpha\in\left[0,\frac{1}{d+1}\right]$.

[By Exercise 4, this guarantees that ρ_α is entangled for $\alpha>\frac{1}{d+1}$. In fact, it is also true that ρ_α is separable for $\alpha\leqslant\frac{1}{d+1}$. But this is harder to prove...]

Exercise 8. *[Werner states]*
The symmetric and anti-symmetric subspaces of $\mathbb{C}^d\otimes\mathbb{C}^d$ are defined, respectively, as the $+1$ and -1 eigenspaces of the flip operator F on $\mathbb{C}^d\otimes\mathbb{C}^d$, as defined in Exercise 5. We thus have
$$S_{\mathbb{C}^d\otimes\mathbb{C}^d}=\left\{\varphi\in\mathbb{C}^d\otimes\mathbb{C}^d : F\varphi=\varphi\right\}\quad \textit{and}\quad A_{\mathbb{C}^d\otimes\mathbb{C}^d}=\left\{\varphi\in\mathbb{C}^d\otimes\mathbb{C}^d : F\varphi=-\varphi\right\}.$$
We denote by Π_S and Π_A the orthogonal projectors onto $S_{\mathbb{C}^d\otimes\mathbb{C}^d}$ and $A_{\mathbb{C}^d\otimes\mathbb{C}^d}$. Those can be written as
$$\Pi_S=\frac{1}{2}(\mathrm{I}+F)\quad \textit{and}\quad \Pi_A=\frac{1}{2}(\mathrm{I}-F).$$
The symmetric and anti-symmetric states on $\mathbb{C}^d\otimes\mathbb{C}^d$ are then defined as $\pi_S=\Pi_S/\mathrm{Tr}(\Pi_S)$ and $\pi_A=\Pi_A/\mathrm{Tr}(\Pi_A)$. Since $\dim(S_{\mathbb{C}^d\otimes\mathbb{C}^d})=d(d+1)/2$ and $\dim(A_{\mathbb{C}^d\otimes\mathbb{C}^d})=d(d-1)/2$, we have
$$\pi_S=\frac{2}{d(d+1)}(\mathrm{I}+F)\quad \textit{and}\quad \pi_A=\frac{2}{d(d-1)}(\mathrm{I}-F).$$
We can now define Werner states on $\mathbb{C}^d\otimes\mathbb{C}^d$: those are states which are convex combinations of the symmetric state π_S and the anti-symmetric state π_A. They have the form
$$\sigma_\lambda=\lambda\pi_S+(1-\lambda)\pi_A,\quad \lambda\in[0,1].$$
Compute $\Gamma(\sigma_\lambda)$ and show that σ_λ is PPT iff $\lambda\in\left[\frac{1}{2},1\right]$.
[By Exercise 4, this guarantees that σ_λ is entangled for $\lambda>\frac{1}{2}$. In fact, it is also true that σ_λ is separable for $\lambda\leqslant\frac{1}{2}$. But this is harder to prove...]

Hints for exercises – October 21st 2025

Hints for exercise 1

1. Write X as a linear combination of product operators.
2. Use the following characterization of positive semi-definite operators: $X \geqslant 0$ iff for all $Y \geqslant 0$, $\mathrm{Tr}(XY) \geqslant 0$.

Hints for exercise 2

Write $\|\varphi - \chi_1 \otimes \chi_2\|^2 = \langle \varphi - \chi_1 \otimes \chi_2 | \varphi - \chi_1 \otimes \chi_2 \rangle$. Then, in order to study the quantity $\langle \varphi | \chi_1 \otimes \chi_2 \rangle$, write φ in its Schmidt decomposition.

Hints for exercise 4

1. Observe that the partial transposition of a product state $\rho_1 \otimes \rho_2$ is positive semi-definite.
2. Write φ in its Schmidt decomposition, and perform the partial transposition of ρ in the Schmidt basis.

Hints for exercise 5

1. What is the action of F on each vector $e_i \otimes e_j$?
2. Observe that $F^2 = \mathrm{I}$.

Hints for exercise 7

Compute $\Gamma(\mathrm{I})$ and $\Gamma(|\psi\rangle\langle\psi|)$.

Hints for exercise 8

Re-write σ_λ as a linear combination of I and F. Then compute $\Gamma(\mathrm{I})$ and $\Gamma(F)$.

Exercise 1

1) Let $X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $Y \in B(\mathcal{H}_1)$. Write $X = \sum_{i=1}^{r} \alpha_i X_1^{(i)} \otimes X_2^{(i)}$, for $X_1^{(i)} \in B(\mathcal{H}_1)$, $X_2^{(i)} \in B(\mathcal{H}_2)$, $\alpha_i \in \mathbb{C}$.

Then: $\mathrm{Tr}_{\mathcal{H}_2}(X) = \sum_{i=1}^{r} \alpha_i \mathrm{Tr}(X_2^{(i)}) X_1^{(i)}$. So: $\mathrm{Tr}(\mathrm{Tr}_{\mathcal{H}_2}(X) Y) = \sum_{i=1}^{r} \alpha_i \mathrm{Tr}(X_2^{(i)}) \mathrm{Tr}(X_1^{(i)} Y)$

And: $\mathrm{Tr}(X(Y \otimes I)) = \sum_{i=1}^{r} \alpha_i \mathrm{Tr}((X_1^{(i)} Y) \otimes (X_2^{(i)} I)) = \sum_{i=1}^{r} \alpha_i \mathrm{Tr}(X_1^{(i)} Y) \mathrm{Tr}(X_2^{(i)})$

2) See lecture notes (Proposition 2.1.8, point (3))

Exercise 2

Let $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ s.t. $\|\psi\| = 1$.

For all $x_1, x_2 \in \mathbb{C}^d$ s.t. $\|x_1\| = \|x_2\| = 1$, we have:

$$\|\psi - x_1 \otimes x_2\|^2 = \langle \psi - x_1 \otimes x_2 | \psi - x_1 \otimes x_2 \rangle = \langle \psi | \psi \rangle + \langle x_1 \otimes x_2 | x_1 \otimes x_2 \rangle - \langle \psi | x_1 \otimes x_2 \rangle - \langle x_1 \otimes x_2 | \psi \rangle$$
$$= \|\psi\|^2 + \|x_1\|^2 \|x_2\|^2 - 2 \mathrm{Re} \langle \psi | x_1 \otimes x_2 \rangle = 2(1 - \mathrm{Re} \langle \psi | x_1 \otimes x_2 \rangle)$$

So minimizing $\|\psi - x_1 \otimes x_2\|$ is equivalent to maximizing $\mathrm{Re}\langle \psi | x_1 \otimes x_2 \rangle$.

Now write $\psi = \sum_{i=1}^{r} \sqrt{\lambda_i}\, u_i \otimes v_i$ in its Schmidt decomposition. We thus have $\mathrm{Re}\langle \psi | x_1 \otimes x_2 \rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, \mathrm{Re}\langle u_i | x_1 \rangle \langle v_i | x_2 \rangle$.

Hence by Hölder inequality: $\mathrm{Re}\langle \psi | x_1 \otimes x_2 \rangle \leqslant \underbrace{\left(\max_{1 \leqslant i \leqslant r} \sqrt{\lambda_i}\right)}_{=\sqrt{\lambda_1}} \left(\sum_{i=1}^{r} \underbrace{|\mathrm{Re}\langle u_i | x_1 \rangle \langle v_i | x_2 \rangle|}_{\leqslant |\langle u_i | x_1 \rangle \langle v_i | x_2 \rangle|}\right) \leqslant \left(\max_{1 \leqslant i \leqslant r} \sqrt{\lambda_i}\right) \underbrace{\left(\sum_{i=1}^{r} |\langle u_i | x_1 \rangle|^2\right)^{1/2}}_{\leqslant \|x_1\| = 1} \underbrace{\left(\sum_{i=1}^{r} |\langle v_i | x_2 \rangle|^2\right)^{1/2}}_{\leqslant \|x_2\| = 1}$

So: $\mathrm{Re}\langle \psi | x_1 \otimes x_2 \rangle \leqslant \sqrt{\lambda_1}$, with equality if $x_1 = u_1$, $x_2 = v_1$.

And therefore: $d(\psi) = \sqrt{2(1 - \sqrt{\lambda_1})} \leqslant \sqrt{2\left(1 - \frac{1}{\sqrt{d}}\right)}$, with equality iff ψ is maximally entangled.

Exercise 3

We have: $|\psi\rangle\langle\psi| = \frac{1}{d} \sum_{i,j=1}^{d} \overbrace{|e_i \otimes e_i \rangle\langle e_j \otimes e_j|}^{= |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j|}$. So for all $X, Y \in B(\mathbb{C}^d)$, we have:

$$\mathrm{Tr}(|\psi\rangle\langle\psi| (X \otimes Y)) = \frac{1}{d} \sum_{i,j=1}^{d} \underbrace{\mathrm{Tr}((|e_i\rangle\langle e_j| X) \otimes (|e_i\rangle\langle e_j| Y))}_{= \mathrm{Tr}(|e_i\rangle\langle e_j| X)\, \mathrm{Tr}(|e_i\rangle\langle e_j| Y) \,=\, \langle e_j | X | e_i \rangle \langle e_j | Y | e_i \rangle} = \frac{1}{d} \sum_{i,j=1}^{d} X_{ij} Y_{ij}$$

and $\mathrm{Tr}(X Y^T) = \sum_{i,j=1}^{d} X_{ij} (Y^T)_{ji} = \sum_{i,j=1}^{d} X_{ij} Y_{ij}$.

Exercise 4

See lecture notes (Theorems 2.3.10 and 2.3.11)

Exercise 5

1) $\forall\, 1 \leqslant i, j \leqslant d$, $F(e_i \otimes e_j) = e_j \otimes e_i$. So $F = \sum_{i,j=1}^{d} |e_i \otimes e_j \rangle\langle e_j \otimes e_i|$

2) It is easy to see that $F^2 = I$. Indeed, for any $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$, writing $\psi = \sum_{i=1}^{r} \alpha_i \psi_1^{(i)} \otimes \psi_2^{(i)}$, we have:

$F\psi = \sum_{i=1}^{r} \alpha_i \psi_2^{(i)} \otimes \psi_1^{(i)}$ and $F^2 \psi = F(F\psi) = \sum_{i=1}^{r} \alpha_i \psi_1^{(i)} \otimes \psi_2^{(i)} = \psi$.

So the eigenvalues of F are $+1$ and -1.

Suppose that $\psi = \sum_{i,j=1}^{d} \psi_{ij}\, e_i \otimes e_j$ is s.t. $F\psi = \psi$ i.e. $\sum_{i,j=1}^{d} \psi_{ij}\, e_j \otimes e_i = \sum_{i,j=1}^{d} \psi_{ij}\, e_i \otimes e_j$.

Then: $\forall\, 1 \leqslant i, j \leqslant d$, $\psi_{ji} = \psi_{ij}$. So $\psi = \sum_{i=1}^{d} \psi_{ii}\, e_i \otimes e_i + \sum_{1 \leqslant i < j \leqslant d} \psi_{ij} (e_i \otimes e_j + e_j \otimes e_i)$

Similarly, if ψ is s.t. $F\psi = -\psi$, then $\forall\, 1 \leqslant i, j \leqslant d$, $\psi_{ji} = -\psi_{ij}$. So $\psi = \sum_{1 \leqslant i < j \leqslant d} \psi_{ij} (e_i \otimes e_j - e_j \otimes e_i)$

Conclusion: The $+1$ eigenspace of F is $\mathrm{span}\left(\{e_i \otimes e_i, 1 \leqslant i \leqslant d\} \cup \{\frac{1}{\sqrt{2}}(e_i \otimes e_j + e_j \otimes e_i), 1 \leqslant i < j \leqslant d\}\right)$ → dimension $\frac{d(d+1)}{2}$

" -1 " " " " $\mathrm{span}\left(\{\frac{1}{\sqrt{2}}(e_i \otimes e_j - e_j \otimes e_i), 1 \leqslant i < j \leqslant d\}\right)$ → dimension $\frac{d(d-1)}{2}$

3) $\forall\, X, Y \in B(\mathbb{C}^d)$, $\mathrm{Tr}(F(X \otimes Y)) = \sum_{i,j=1}^{d} \mathrm{Tr}(|e_i\rangle\langle e_j| X)\, \mathrm{Tr}(|e_j\rangle\langle e_i| Y) = \sum_{i,j=1}^{d} X_{ij} Y_{ji} = \mathrm{Tr}(XY)$

Exercise 6

$$\Gamma(|\psi\rangle\langle\psi|) = \frac{1}{d}\sum_{i,j=1}^{d} \underbrace{\Gamma(|e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j|)}_{= T(|e_i\rangle\langle e_j|)\otimes|e_i\rangle\langle e_j|} = \frac{1}{d}\sum_{i,j=1}^{d} |e_j\rangle\langle e_i| \otimes |e_i\rangle\langle e_j| = \frac{1}{d} F.$$

Exercise 7

We have $\Gamma(|\psi\rangle\langle\psi|) = \frac{1}{d} F$ and $\Gamma(I) = I$. So $\Gamma(\rho_\alpha) = \alpha \frac{F}{d} + (1-\alpha)\frac{I}{d^2}$

Hence, the eigenvalues of $\Gamma(\rho_\alpha)$ are $\lambda_1 = \frac{1-\alpha}{d^2} + \frac{\alpha}{d}$ (with multiplicity $\frac{d(d+1)}{2}$) and $\lambda_2 = \frac{1-\alpha}{d^2} - \frac{\alpha}{d}$ (with multiplicity $\frac{d(d-1)}{2}$)

$\lambda_1 \geq 0$ for all $0 \leq \alpha \leq 1$ and $\lambda_2 \geq 0$ iff $0 \leq \alpha \leq \frac{1}{d+1}$.

So ρ_α is PPT iff $0 \leq \alpha \leq \frac{1}{d+1}$

Exercise 8

First observe that: $\sigma_\lambda = \frac{2\lambda}{d(d+1)}(I+F) + \frac{2(1-\lambda)}{d(d-1)}(I-F) = \frac{2}{d(d^2-1)}\left((d+1-2\lambda)I + ((2\lambda-1)d-1)F\right)$

Next we have $\Gamma(F) = d|\psi\rangle\langle\psi|$ and $\Gamma(I) = I$. So $\Gamma(\sigma_\lambda) = \frac{2}{d(d^2-1)}\underbrace{\left((d+1-2\lambda)I + d((2\lambda-1)d-1)|\psi\rangle\langle\psi|\right)}_{= X_\lambda}$

Hence, the eigenvalues of X_λ are $\lambda_1 = d+1-2\lambda$ (with multiplicity d^2-1) and $\lambda_2 = (d+1-2\lambda) + d((2\lambda-1)d-1) = (2\lambda-1)(d^2-1)$ (with multiplicity 1)

$\lambda_1 \geq 0$ for all $0 \leq \lambda \leq 1$ and $\lambda_2 \geq 0$ iff $0 \leq \lambda \leq \frac{1}{2}$.

So σ_λ is PPT iff $0 \leq \lambda \leq \frac{1}{2}$.

Exercises – November 5th 2025

Exercise 1. *Find the unital completely positive linear maps that are the dual of the following trace-preserving completely positive linear maps:*

1. $\Phi : X \in B(\mathcal{H}) \mapsto \mathrm{Tr}(X)\sigma \in B(\mathcal{H})$, *where* $\sigma \in \mathrm{D}(\mathcal{H})$.
2. $\Phi : X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \mapsto \mathrm{Tr}_{\mathcal{H}_2}(X) \in B(\mathcal{H}_1)$.
3. $\Phi : X \in B(\mathcal{H}_1) \mapsto X \otimes \sigma \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, *where* $\sigma \in \mathrm{D}(\mathcal{H}_2)$.

Exercise 2. *Define* Π_λ *as the depolarizing channel with parameter* $\lambda \in [0,1]$ *on* $\mathcal{M}_d(\mathbb{C})$, *i.e.*

$$\Pi_\lambda : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \lambda X + (1-\lambda)\,\mathrm{Tr}(X)\frac{\mathrm{I}}{d} \in \mathcal{M}_d(\mathbb{C}).$$

Show that the Choi matrix of Π_λ *is* $C(\Pi_\lambda) = d\rho_\lambda$, *where* ρ_λ *is the so-called isotropic state with parameter* $\lambda \in [0,1]$ *on* $\mathbb{C}^d \otimes \mathbb{C}^d$, *i.e.*

$$\rho_\lambda = \lambda |\psi\rangle\langle\psi| + (1-\lambda)\frac{\mathrm{I}}{d^2},$$

with $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ *the maximally entangled unit vector.*

Exercise 3. *Let* ρ *be a state on* $\mathbb{C}^d$ *and define the quantum channel* Π_ρ *on* $\mathcal{M}_d(\mathbb{C})$ *as*

$$\Pi_\rho : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \mathrm{Tr}(X)\rho \in \mathcal{M}_d(\mathbb{C}).$$

1. *Compute* $C(\Pi_\rho)$, *the Choi matrix of* Π_ρ.
2. *Deduce from that what is the Kraus rank of* Π_ρ.

Exercise 4. *Define the fully dephasing channel* Δ *on* $\mathcal{M}_d(\mathbb{C})$ *and the fully depolarizing channel* Π *on* $\mathcal{M}_d(\mathbb{C})$ *as*

$$\Delta : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{i=1}^d \langle e_i|X|e_i\rangle |e_i\rangle\langle e_i| \in \mathcal{M}_d(\mathbb{C}) \quad \text{and} \quad \Pi : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \mathrm{Tr}(X)\frac{\mathrm{I}}{d} \in \mathcal{M}_d(\mathbb{C}),$$

where $\{e_1, \ldots, e_d\}$ *is an orthonormal basis of* $\mathbb{C}^d$.

1. *Write Kraus decompositions of* Δ *and* Π *in terms of Kraus operators of the form* $E_{ij} = |e_i\rangle\langle e_j|$, $1 \leqslant i,j \leqslant d$.
2. *Define the unitary operators* U *and* V *on* $\mathbb{C}^d$ *as*

$$U = \sum_{j=1}^d \omega^j |e_j\rangle\langle e_j|, \text{ where } \omega = e^{2i\pi/d} \quad \text{and} \quad V = \sum_{j=1}^d |e_{j+1}\rangle\langle e_j|, \text{ where by convention } e_{d+1} = e_1.$$

   Show that Δ *and* Π *have the following Kraus decompositions*

$$\Delta : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \frac{1}{d}\sum_{p=1}^d U^p X U^{*p} \in \mathcal{M}_d(\mathbb{C}) \quad \text{and} \quad \Pi : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \frac{1}{d^2}\sum_{p,q=1}^d U^p V^q X V^{*q} U^{*p} \in \mathcal{M}_d(\mathbb{C}).$$

   [This proves that Δ *and* Π *are in fact mixtures of unitary channels.]*

Exercise 5. *Let* $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ *and* $\Psi : B(\mathcal{H}_1') \to B(\mathcal{H}_2')$ *be linear maps. Show that, if* Φ *and* Ψ *are completely positive, then so are* $\Psi \circ \Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2')$ *(assuming that* $\mathcal{H}_1' = \mathcal{H}_2$, *so that the composition is well-defined) and* $\Phi \otimes \Psi : B(\mathcal{H}_1 \otimes \mathcal{H}_1') \to B(\mathcal{H}_2 \otimes \mathcal{H}_2')$.

Exercise 6. *Let* $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ *and* $\Psi : B(\mathcal{K}_1) \to B(\mathcal{H}_2)$ *be quantum channels. Show that, if one of them is entanglement-breaking, then, for any state* ρ *on* $\mathcal{H}_1 \otimes \mathcal{K}_1$, $\Phi \otimes \Psi(\rho)$ *is a separable state on* $\mathcal{H}_2 \otimes \mathcal{K}_2$.

Exercise 7.

1. *Let* $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ *be a linear map that preserves Hermiticity (i.e. for all* $X \in B(\mathcal{H}_1)$, *if* $X^* = X$ *then* $\Phi(X)^* = \Phi(X)$*). We admit that this is equivalent to* $C(\Phi) \in B(\mathcal{H}_2 \otimes \mathcal{H}_1)$ *being Hermitian. Show that* Φ *can be written as the difference of two completely positive linear maps.*
2. *Let* $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ *be a linear map. Show that* Φ *can be written as the linear combination of four completely positive linear maps.*

Hints for exercises – November 5th 2025

Hints for exercise 1

Start from the duality relation, for all X, Y, $\mathrm{Tr}(X\Phi^*(Y)) = \mathrm{Tr}(\Phi(X)Y)$. Then, for question (1) take $X = |e_i\rangle\langle e_j|$, $1 \leqslant i, j \leqslant \dim(\mathcal{H})$. While for questions (2) and (3) rewrite $\mathrm{Tr}(\Phi(X)Y)$ as $\mathrm{Tr}(XZ_{Y,\Phi})$.

Hints for exercise 2

Fix $\{e_1, \dots, e_d\}$ an orthonormal basis of $\mathbb{C}^d$ and write the expressions of $C(\Pi_\lambda)$ and ρ_λ in the product orthonormal basis of $\mathbb{C}^d \otimes \mathbb{C}^d$ $\{e_i \otimes e_j,\ 1 \leqslant i, j \leqslant d\}$.

Hints for exercise 3

For question (2), recall that the Kraus rank of Π_ρ is the rank of $C(\Pi_\rho)$.

Hints for exercise 4

1. For Π, write the trace as a contraction with some E_{ij}'s and the identity as a sum of some E_{ij}'s.
2. First, write the expression of U^p, V^q, U^pV^q. Second, given $n \in \{-d, -(d-1), \dots, d-1, d\} \setminus \{0\}$, what is the value of $\sum_{p=1}^d e^{2i\pi np/d}$?

Hints for exercise 5

For $\Phi \otimes \Psi$, observe that it can be written as $(\mathrm{id} \otimes \Psi) \circ (\Phi \otimes \mathrm{id})$.

Hints for exercise 6

First observe that $\Phi \otimes \Psi = (\mathrm{id} \otimes \Psi) \circ (\Phi \otimes \mathrm{id}) = (\Phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi)$. And second observe that, for any quantum channel Θ and any separable state σ, $\Theta \otimes \mathrm{id}(\sigma)$ is a separable state.

Hints for exercise 7

1. Observe that any Hermitian matrix C can be written as the difference of two positive semidefinite matrices: $C = C^+ - C^-$, where $C^+, C^- \geqslant 0$.
2. Observe that any matrix C can be written as the linear combination of two Hermitian matrices: $C = C_R + iC_I$, where $C_R^* = C_R, C_I^* = C_I$.

Exercise 1

1) $\forall\, X, Y \in B(\mathcal{H})$, $\mathrm{Tr}(X\,\Phi^*(Y)) = \mathrm{Tr}(\Phi(X)\,Y) = \mathrm{Tr}(X)\,\mathrm{Tr}(\sigma Y)$

Taking $X = |e_i\rangle\langle e_j|$, for $1 \leq i,j \leq d$, we get: $\forall\, Y \in B(\mathcal{H})$, $\langle e_j|\Phi^*(Y)|e_i\rangle = \overbrace{\langle e_j|e_i\rangle}^{\delta_{ij}}\,\mathrm{Tr}(\sigma Y)$

Hence: $\forall\, Y \in B(\mathcal{H})$, $\Phi^*(Y) = \sum_{i,j=1}^{d} \langle e_j|\Phi^*(Y)|e_i\rangle\, |e_j\rangle\langle e_i| = \sum_{i=1}^{d} \mathrm{Tr}(\sigma Y)\,|e_i\rangle\langle e_i| = \mathrm{Tr}(\sigma Y)\, I$

2) $\forall\, X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $Y \in B(\mathcal{H}_1)$, $\mathrm{Tr}(X\,\Phi^*(Y)) = \mathrm{Tr}(\Phi(X)\,Y) = \mathrm{Tr}(\mathrm{Tr}_{\mathcal{H}_2}(X)\,Y) = \mathrm{Tr}(X\,(Y \otimes I))$

Hence: $\forall\, Y \in B(\mathcal{H}_1)$, $\Phi^*(Y) = Y \otimes I$.

3) $\forall\, X \in B(\mathcal{H}_1)$, $Y \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\mathrm{Tr}(X\,\Phi^*(Y)) = \mathrm{Tr}(\Phi(X)\,Y) = \mathrm{Tr}((X \otimes \sigma)\,Y) = \mathrm{Tr}((X \otimes I)(I \otimes \sigma)\,Y)$
$= \mathrm{Tr}(X\, \mathrm{Tr}_{\mathcal{H}_2}((I \otimes \sigma)\,Y))$

Hence: $\forall\, Y \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\Phi^*(Y) = \mathrm{Tr}_{\mathcal{H}_2}((I \otimes \sigma)\,Y)$

Exercise 2

$$C(\Pi_\lambda) = \sum_{i,j=1}^{d} \Pi_\lambda(|e_i\rangle\langle e_j|) \otimes |e_i\rangle\langle e_j| = \sum_{i,j=1}^{d} \Big(\lambda\,|e_i\rangle\langle e_j| + (1-\lambda)\,\overbrace{\langle e_j|e_i\rangle}^{=\delta_{ij}}\,\frac{I}{d}\Big) \otimes |e_i\rangle\langle e_j|$$
$$= \lambda \sum_{i,j=1}^{d} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| + (1-\lambda)\sum_{i=1}^{d} \frac{I}{d} \otimes |e_i\rangle\langle e_i| = \lambda\, d\, |\psi\rangle\langle\psi| + (1-\lambda)\,\frac{I}{d} \otimes I = d\Big(\lambda\,|\psi\rangle\langle\psi| + (1-\lambda)\,\frac{I}{d^2}\Big)$$

Hence: $C(\Pi_\lambda) = d\,\rho_\lambda$.

Exercise 3

1) $C(\Pi_\rho) = \sum_{i,j=1}^{d} \Pi_\rho(|e_i\rangle\langle e_j|) \otimes |e_i\rangle\langle e_j| = \sum_{i,j=1}^{d} (\overbrace{\langle e_j|e_i\rangle}^{\delta_{ij}}\,\rho) \otimes |e_i\rangle\langle e_j| = \sum_{i=1}^{d} \rho \otimes |e_i\rangle\langle e_i| = \rho \otimes I$

2) The Kraus rank of Π_ρ is the rank of $C(\Pi_\rho) = \rho \otimes I$, i.e. $\mathrm{rank}(\rho) \times \mathrm{rank}(I) = \mathrm{rank}(\rho) \times d$.

Exercise 4

1) $\Delta(X) = \sum_{i=1}^{d} |e_i\rangle\langle e_i|X|e_i\rangle\langle e_i| = \sum_{i=1}^{d} E_{ii}\, X\, E_{ii}^*$

$\Pi(X) = \sum_{j=1}^{d} \langle e_j|X|e_j\rangle \sum_{i=1}^{d} \frac{1}{d}\,|e_i\rangle\langle e_i| = \frac{1}{d}\sum_{i,j=1}^{d} |e_i\rangle\langle e_j|X|e_j\rangle\langle e_i| = \frac{1}{d}\sum_{i,j=1}^{d} E_{ij}\, X\, E_{ij}^*$

2) First observe that: $\forall\, 1 \leq p \leq d$, $U^p = \sum_{j=1}^{d} \omega^{jp}\,|e_j\rangle\langle e_j|$.

Hence: $\frac{1}{d}\sum_{p=1}^{d} U^p X U^{*p} = \frac{1}{d}\sum_{p=1}^{d}\sum_{j,k=1}^{d} \omega^{jp}\,\bar{\omega}^{kp}\,|e_j\rangle\langle e_j|X|e_k\rangle\langle e_k| = \frac{1}{d}\sum_{j,k=1}^{d}\Big(\sum_{p=1}^{d}\omega^{jp}\,\bar{\omega}^{kp}\Big)\,|e_j\rangle\langle e_j|X|e_k\rangle\langle e_k|$

Now, $\omega^{jp}\,\bar{\omega}^{kp} = e^{2i\pi(j-k)p/d}$. So $\sum_{p=1}^{d} \omega^{jp}\,\bar{\omega}^{kp} = \begin{cases} 0 & \text{if } k \neq j \\ d & \text{if } k = j \end{cases}$

And thus: $\frac{1}{d}\sum_{p=1}^{d} U^p X U^{*p} = \sum_{j=1}^{d} |e_j\rangle\langle e_j|X|e_j\rangle\langle e_j| = \Delta(X)$

Observe next that: $\forall\, 1 \leq q \leq d$, $V^q = \sum_{j=1}^{d} |e_{j+q}\rangle\langle e_j|$. So $\forall\, 1 \leq p,q \leq d$, $U^p V^q = \sum_{j=1}^{d} \omega^{jp}\,|e_j\rangle\langle e_{j-q}|$

Hence: $\frac{1}{d^2}\sum_{p,q=1}^{d} U^p V^q X V^{*q} U^{*p} = \frac{1}{d^2}\sum_{p,q=1}^{d}\sum_{j,k=1}^{d} \omega^{jp}\,\bar{\omega}^{kp}\,|e_j\rangle\langle e_{j-q}|X|e_{k-q}\rangle\langle e_k|$

Now, as before: $\sum_{p=1}^{d} \omega^{jp}\,\bar{\omega}^{kp} = \begin{cases} 0 & \text{if } k \neq j \\ d & \text{if } k = j \end{cases}$

And thus: $\frac{1}{d^2}\sum_{p,q=1}^{d} U^p V^q X V^{*q} U^{*p} = \frac{1}{d}\sum_{q=1}^{d}\sum_{j=1}^{d} |e_j\rangle\langle e_{j-q}|X|e_{j-q}\rangle\langle e_j| = \frac{1}{d}\sum_{j,k=1}^{d} |e_j\rangle\langle e_k|X|e_k\rangle\langle e_j| = \Pi(X)$

(↑ change of variable $k = j - q$)

Exercise 5

- For any Hilbert space $\mathcal{K}$, $(\Psi\circ\Phi)\otimes id : B(\mathcal{H}_1\otimes\mathcal{K}) \to B(\mathcal{H}_2'\otimes\mathcal{K})$ can be written as $(\Psi\otimes id)\circ(\Phi\otimes id)$.
  Hence, for all $X \geqslant 0$, $(\Psi\circ\Phi)\otimes id\,(X) = \underbrace{\Psi\otimes id\,(\underbrace{\Phi\otimes id(X)}_{\geqslant 0 \text{ because } \Phi \text{ CP}})}_{\geqslant 0 \text{ because } \Psi \text{ CP}} \geqslant 0$. So $\Psi\circ\Phi$ is CP

- Observe that $\Phi\otimes\Psi = (id\otimes\Psi)\circ(\Phi\otimes id)$.
  So for any Hilbert space $\mathcal{K}$, $\Phi\otimes\Psi\otimes id : B(\mathcal{H}_1\otimes\mathcal{H}_1'\otimes\mathcal{K}) \to B(\mathcal{H}_2\otimes\mathcal{H}_2'\otimes\mathcal{K})$ can be written as $(id\otimes\Psi\otimes id)\circ(\Phi\otimes id\otimes id)$
  Hence, for all $X\geqslant 0$, $\Phi\otimes\Psi\otimes id\,(X) = \underbrace{id\otimes\Psi\otimes id\,(\underbrace{\Phi\otimes id\otimes id\,(X)}_{\geqslant 0 \text{ because } \Phi \text{ CP}})}_{\geqslant 0 \text{ because } \Psi \text{ CP}}$. So $\Phi\otimes\Psi$ is CP

Exercise 6

Observe that $\Phi\otimes\Psi = (id\otimes\Psi)\circ(\Phi\otimes id) = (\Phi\otimes id)\circ(id\otimes\Psi)$

Suppose that Φ is entanglement breaking. Then for any $\varrho\in D(\mathcal{H}_1\otimes\mathcal{K}_1)$, $\Phi\otimes id(\varrho)\in S(\mathcal{H}_2\otimes\mathcal{K}_1)$, and thus $id\otimes\Psi\,(\Phi\otimes id(\varrho))\in S(\mathcal{H}_2\otimes\mathcal{K}_2)$.

Similarly, if Ψ is entanglement-breaking, then for any $\varrho\in D(\mathcal{H}_1\otimes\mathcal{K}_1)$, $id\otimes\Psi(\varrho)\in S(\mathcal{H}_1\otimes\mathcal{K}_2)$, and thus $\Phi\otimes id\,(id\otimes\Psi(\varrho))\in S(\mathcal{H}_2\otimes\mathcal{K}_2)$

[We also used the fact that if Φ is CP (in fact even just positive) and TP, then, for any separable state σ, $\Phi\otimes id(\sigma)$ is a separable state. Indeed: $\Phi\otimes id\left(\sum_{i=1}^{r}\lambda_i\sigma_1^{(i)}\otimes\sigma_2^{(i)}\right) = \sum_{i=1}^{r}\lambda_i\underbrace{\Phi(\sigma_1^{(i)})}_{\text{state}}\otimes\sigma_2^{(i)}$.]

Exercise 7

1) Since $C(\Phi)$ is Hermitian, it can be written as the difference of two positive semidefinite operators: $C(\Phi) = C^+ - C^-$, where $C^+\geqslant 0$, $C^-\geqslant 0$ (Indeed: $C(\Phi) = \sum_i \lambda_i |u_i\rangle\langle u_i|$ and $C^+ = \sum_{i \text{ s.t. } \lambda_i>0}\lambda_i|u_i\rangle\langle u_i|$, $C^- = \sum_{i \text{ s.t. } \lambda_i<0} -\lambda_i|u_i\rangle\langle u_i|$)
   Since $C^+, C^-\geqslant 0$, there exist CP linear maps Φ^+,Φ^- s.t. $C^+ = C(\Phi^+)$, $C^- = C(\Phi^-)$.
   By linearity, we have $\Phi = \Phi^+ - \Phi^-$.

2) Write $C(\Phi) = C_R + i\,C_I$, where $C_R = \frac{C+C^*}{2}$, $C_I = -i\,\frac{C-C^*}{2}$ are Hermitian.
   Hence $C_R = C_R^+ - C_R^-$ and $C_I = C_I^+ - C_I^-$, where $C_R^+, C_R^-; C_I^+, C_I^- \geqslant 0$
   So there exist CP linear maps $\Phi_R^+,\Phi_R^-,\Phi_I^+,\Phi_I^-$ s.t. $C_R^+ = C(\Phi_R^+)$, $C_R^- = C(\Phi_R^-)$, $C_I^+ = C(\Phi_I^+)$, $C_I^- = C(\Phi_I^-)$.
   And by linearity, we have $\Phi = \Phi_R^+ - \Phi_R^- + i\,\Phi_I^+ - i\,\Phi_I^-$