Need quantum entanglement? Don't sweat it, just pick at random!

Entangling quantum information theory and asymptotic geometric analysis

Cécilia Lancien

Institut Fourier & CNRS

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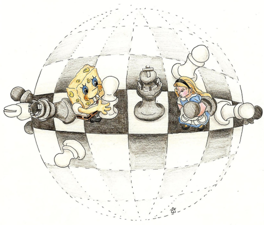

Outline

What is quantum physics and why the need for it?

Quantum physics: theory that progressively arose to *explain certain observed physical phenomena*, which classical physics could not account for.

Examples of puzzling phenomena at the origin of the theory (in the 1900's):
• black-body radiation: resolved by *quantization* of light into photons (Planck, Einstein)
• electron interference: resolved by *wave function* description of particles (de Broglie, Schrödinger)

Full development of the theory from the 1920's (Heisenberg, Born, Dirac, Bohr, von Neumann, Bell, etc.)

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Some key aspects of quantum physics:

- Not needed to describe ordinary scale behaviors, only *very small scale* ones.
  ↳ classical physics: approximation of quantum physics valid at macroscopic scale
- *Inherently probabilistic*, not just as a consequence of imprecisions or lack of knowledge.
  ↳ even in ideal scenario, measurement outcomes cannot be predicted with certainty
- Neat mathematical formalism but *not so physically intuitive*.
  ↳ striking consequences: uncertainty, superposition, entanglement, etc.

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Goal of this talk: Explain how *asymptotic geometric analysis* can be useful to tackle problems arising in *quantum information theory*.
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Question 1: What is quantum information theory?

→ communication, computation, encryption, etc.

Understanding if some *information processing tasks* could be performed more efficiently if the physical support of information obeys the laws of *quantum rather than classical physics*.

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More specifically: Quantify the potential advantage of quantum over classical resources.

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Question 2: What is asymptotic geometric analysis?

Using *probabilistic techniques* to study *Banach spaces of finite but high dimension*.

What is this talk about? (continued)

Why "high dimension"? 2-level quantum system (e.g. electron's spin, photon's polarization)
Quantum system composed of 1 qubit: described by the space $\mathbf{C}^2$.
Quantum system composed of M qubits: described by the space $(\mathbf{C}^2)^{\otimes M} \equiv \mathbf{C}^{2^M}$.
$\longrightarrow$ The dimension of a system *grows exponentially* with its number of subsystems, so we have to deal with high dimension as soon as more than a few qubits are involved.

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Why "probabilistic techniques"?

1. Identify the *typical properties of quantum systems*, i.e. when they are picked at random.
2. Prove the *existence of quantum systems having certain properties*, using random constructions.

Note: "typical" $=$ "with probability going to 1 as the underlying dimension grows".
$\longrightarrow$ There are usually two steps in the argument:

1. Identify the average behavior of the property under consideration.
2. Show that this average behavior is generic in high dimension.
   Concentration of measure phenomenon: a sufficiently 'well-behaved' many-variable function has a very small probability of deviating from its average.

Mathematical formalism of quantum physics in a nutshell

- Quantum system composed of 1 subsystem: Complex Hilbert space H.
$\longrightarrow$ Finite-dimensional case: $\mathrm{H} \equiv \mathbf{C}^d$ for some $d \in \mathbf{N}$ (equipped with inner product $\langle \cdot | \cdot \rangle$).

  Quantum system composed of M subsystems: Complex Hilbert space $\mathrm{H} = \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$, tensor product of the complex Hilbert spaces corresponding to each subsystem.
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- State of system H:
  Pure state: unit vector ψ in H (or rather associated rank 1 projector $\psi\psi^*$ on H).
  Mixed state: convex combination of pure states

$$\rho = \sum_{k=1}^{r} \mu_k \psi_k \psi_k^*, \text{ with } \begin{cases} \mu_1, \ldots, \mu_r \geqslant 0, \ \sum_{k=1}^{r} \mu_k = 1 \\ \psi_1, \ldots, \psi_r \text{ unit vectors in } \mathrm{H} \end{cases} .$$

  Equivalently, ρ is a self-adjoint positive semidefinite and trace 1 operator on H.
  $\longrightarrow$ Finite-dimensional case: $\rho \in \mathcal{M}_d(\mathbf{C})$ s.t. $\rho^* = \rho$, $\rho \geqslant 0$ and $\mathrm{Tr}(\rho) = 1$.

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Classical-Quantum correspondence:
Given a quantum state $\rho \in \mathcal{M}_d(\mathbf{C})$, $\mathrm{spec}(\rho) \in \mathbf{R}^d$ is a probability vector, i.e. a classical state.
Extra freedom: choice of basis where ρ is diagonal.

Definition [Separability and entanglement]

A state ρ on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ is called *separable* if it is a convex combination of product states, i.e.

└→ self-adjoint positive semidefinite operator with trace 1 on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$

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Otherwise it is called *entangled*.

[Note: If ρ is a pure state, i.e. $\rho = \psi\psi^*$ for some unit vector $\psi \in \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$, then ρ is separable iff $\psi = \psi_1 \otimes \cdots \otimes \psi_M$ for some unit vectors $\psi_i \in \mathrm{H}_i$, for each $1 \leqslant i \leqslant M$.]

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Observation:

- In a *product state*, there are *no correlations* between subsystems.
  $\hookrightarrow$ $\rho = \rho_1 \otimes \cdots \otimes \rho_M$ $\hookrightarrow$ cf. independent random variables: joint distribution factorizes
- In a *separable state*, there are only *classical correlations* between subsystems.
  $\hookrightarrow$ $\rho = \sum_{k=1}^r \mu_k \rho_1^k \otimes \cdots \otimes \rho_M^k$ $\hookrightarrow$ described by the probability distribution $\mu = (\mu_1, \ldots, \mu_r)$
- In an *entangled state*, there are *intrinsically quantum correlations* between subsystems.

Fact: In most information processing tasks, a quantum system must be in an entangled state to provide an advantage over a classical system.
$\longrightarrow$ *Certifying entanglement* (and quantifying it) is an important issue in practice.

Outline

Tensor norms in Banach spaces

Let $A_1, \ldots, A_M$ be Banach spaces. Given $x \in A_1 \otimes \cdots \otimes A_M$, its *injective norm* is

$$\|x\|_{A_1 \otimes_\varepsilon \cdots \otimes_\varepsilon A_M} := \sup\left\{ \left| \langle\, b_1 \otimes \cdots \otimes b_M \,|\, x \,\rangle \right| \,:\, b_i \in A_i^*,\ \|b_i\|_{A_i^*} \leqslant 1 \right\},$$

and its *projective norm* is

$$\|x\|_{A_1 \otimes_\pi \cdots \otimes_\pi A_M} := \inf\left\{ \sum_{k=1}^{r} |\alpha_k| \,:\, a_i^k \in A_i,\ \|a_i^k\|_{A_i} \leqslant 1,\ x = \sum_{k=1}^{r} \alpha_k\, a_1^k \otimes \cdots \otimes a_M^k,\ r \in \mathbf{N} \right\}.$$

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These norms are dual to one another: for all $x \in A_1 \otimes \cdots \otimes A_M$,

$$\|x\|_{A_1 \otimes_\varepsilon \cdots \otimes_\varepsilon A_M} = \sup \left\{ |\langle y \,|\, x \rangle| \;:\; \|y\|_{A_1^* \otimes_\pi \cdots \otimes_\pi A_M^*} \leqslant 1 \right\},$$

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The injective and projective norms are examples of *tensor norms*: for all $a_1\in A_1,\dots,a_M\in A_M$,

$$\|a_1\otimes\cdots\otimes a_M\|_{A_1\otimes_\varepsilon\cdots\otimes_\varepsilon A_M}=\|a_1\otimes\cdots\otimes a_M\|_{A_1\otimes_\pi\cdots\otimes_\pi A_M}=\|a_1\|_{A_1}\cdots\|a_M\|_{A_M}.$$

And they are *extremal* among such norms: for any other tensor norm $\|\cdot\|$ on $A_1\otimes\cdots\otimes A_M$,

$$\|\cdot\|_{A_1\otimes_\varepsilon\cdots\otimes_\varepsilon A_M}\leqslant\|\cdot\|\leqslant\|\cdot\|_{A_1\otimes_\pi\cdots\otimes_\pi A_M}.$$

- **Pure state entanglement:**

Banach spaces $\left(\mathbf{C}^{d_i}, \|\cdot\|_{\ell_2^{d_i}}\right)$, $1 \leqslant i \leqslant M$. $\left[\text{Notation: } \forall\, x \in \mathbf{C}^d,\ \|x\|_{\ell_2^d} := \left(\sum_{k=1}^d |x_k|^2\right)^{1/2}\right]$ ← vector 2-norm

A pure state $\psi \in \mathbf{C}^{d_1} \otimes \cdots \otimes \mathbf{C}^{d_M}$ is s.t. $\|\psi\|_{\ell_2^{d_1 \cdots d_M}} = 1$, and thus $\begin{cases} \|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_M}} \leqslant 1 \\ \|\psi\|_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_M}} \geqslant 1 \end{cases}$.

because $\|\cdot\|_{\ell_2^{d_1 \cdots d_M}}$ is a tensor norm

And ψ is separable iff $\|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_M}} = \|\psi\|_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_M}} = 1$.

Reminder: $\|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_M}} := \sup\left\{ |\langle \varphi_1 \otimes \cdots \otimes \varphi_M | \psi \rangle| \,:\, \varphi_i \in \mathbf{C}^{d_i},\ \|\varphi_i\|_{\ell_2^{d_i}} = 1 \right\}$

$$\|\psi\|_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_M}} := \inf\left\{ \sum_{k=1}^r |\alpha_k| \,:\, \chi_i^k \in \mathbf{C}^{d_i},\ \|\chi_i^k\|_{\ell_2^{d_i}} = 1,\ \psi = \sum_{k=1}^r \alpha_k \chi_1^k \otimes \cdots \otimes \chi_M^k \right\}$$

• Pure state entanglement:

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• Mixed state entanglement:

matrix 1-norm

Banach spaces $\left(\mathcal{M}_{d_i}(\mathbf{C}), \|\cdot\|_{S_1^{d_i}}\right)$, $1 \leqslant i \leqslant M$. $\Big[$ Notation: $\forall\, X \in \mathcal{M}_d(\mathbf{C})$, $\|X\|_{S_1^d} := \mathrm{Tr}\,|X|$ $\Big]$

because $\|\cdot\|_{S_1^{d_1\cdots d_M}}$ is a tensor norm

A mixed state $\rho \in \mathcal{M}_{d_1}(\mathbf{C}) \otimes \cdots \otimes \mathcal{M}_{d_M}(\mathbf{C})$ is s.t. $\|\rho\|_{S_1^{d_1\cdots d_m}} = 1$, and thus $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_M}} \geqslant 1$.

And ρ is separable iff $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_M}} = 1$.

Reminder: $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_M}} := \inf\left\{ \sum_{k=1}^r |\alpha_k| \,:\, \tau_i^k \in \mathcal{M}_{d_i}(\mathbf{C}),\ \|\tau_i^k\|_{S_1^{d_i}} = 1,\ \rho = \sum_{k=1}^r \alpha_k\, \tau_1^k \otimes \cdots \otimes \tau_M^k \right\}$

If $\|\psi\|_{\ell_2^{d_1}\otimes_\varepsilon\cdots\otimes_\varepsilon\ell_2^{d_M}} \ll 1$ or $\|\psi\|_{\ell_2^{d_1}\otimes_\pi\cdots\otimes_\pi\ell_2^{d_M}} \gg 1$, then ψ is ‘very’ entangled.

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Questions: Can this be made *quantitative*?
Is there an *operational interpretation* for these norms being ‘small’ or ‘large’?

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Definition [Geometric measure of entanglement (Wei/Goldbart)]

Let $\psi \in \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ be a pure state. Its *geometric measure of entanglement (GME)* is

$$E(\psi) := -\log\sup\left\{ |\langle \varphi_1 \otimes \cdots \otimes \varphi_M|\psi\rangle|^2 \,:\, \varphi_i \in \mathrm{H}_i,\ \|\varphi_i\|_{\ell_2^{d_i}} = 1 \right\} = -2\log\|\psi\|_{\ell_2^{d_1}\otimes_\varepsilon\cdots\otimes_\varepsilon\ell_2^{d_M}}.$$

Fact: The GME has a meaning in several quantum information processing tasks (construction of *entanglement witnesses*, *discrimination of quantum states* under local operations, detection of *phase transition* in quantum many-body systems, etc.)

From the definition, $E(\psi) = 0$ iff ψ is separable. But how large can $E(\psi)$ be for ψ entangled?
$\hookrightarrow$ *faithful entanglement measure* for multipartite pure states

Let us look at the previous definitions in the case of pure states for $M=2$ and $\mathrm{H}_1 \equiv \mathrm{H}_2 \equiv \mathbf{C}^d$.

We can identify $x = \sum_{k,l=1}^{d} x_{kl}\, e_k \otimes e_l \in \mathbf{C}^d \otimes \mathbf{C}^d$ with $X = \sum_{k,l=1}^{d} x_{kl}\, e_k e_l^* \in \mathcal{M}_d(\mathbf{C})$.

The *singular value decomposition* of X thus corresponds to the *Schmidt decomposition* of x:

$$X = \sum_{k=1}^{r} s_k\, u_k v_k^* \longleftrightarrow x = \sum_{k=1}^{r} s_k\, u_k \otimes v_k, \text{ with } r \leqslant d \text{ and } \begin{cases} s_1, \ldots, s_r > 0 \\ \{u_k\}_{k=1}^r, \{v_k\}_{k=1}^r \text{ o.n.f. in } \mathbf{C}^d \end{cases} .$$

This identification preserves the Euclidean norm: $\|x\|_{\ell_2^{d^2}} = \left(\sum_{k=1}^r s_k^2\right)^{1/2} = \|X\|_{S_2^d}$.

While $\|x\|_{\ell_2^d \otimes_\varepsilon \ell_2^d} = \max_{1\leqslant k\leqslant r} s_k = \|X\|_{S_\infty^d}$ and $\|x\|_{\ell_2^d \otimes_\pi \ell_2^d} = \sum_{k=1}^r s_k = \|X\|_{S_1^d}$.

$\longrightarrow$ If $\|x\|_{\ell_2^{d^2}} = 1$, then by Cauchy-Schwarz $\frac{1}{\sqrt{d}} \leqslant \|x\|_{\ell_2^d \otimes_\varepsilon \ell_2^d} \leqslant 1$ and $1 \leqslant \|x\|_{\ell_2^d \otimes_\pi \ell_2^d} \leqslant \sqrt{d}$.

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More generally: Assume that $d_1 \leqslant \cdots \leqslant d_M$ and set $D := d_1 \times \cdots \times d_{M-1}$.

- For any pure state $\psi \in \mathbf{C}^{d_1} \otimes \cdots \otimes \mathbf{C}^{d_M}$, $\|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_M}} \geqslant \frac{1}{\sqrt{D}}$ and $\|\psi\|_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_M}} \leqslant \sqrt{D}$.

Proof idea: Recursive argument from bipartite case.

- For any mixed state $\rho \in \mathcal{M}_{d_1}(\mathbf{C}) \otimes \cdots \otimes \mathcal{M}_{d_M}(\mathbf{C})$, $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_M}} \leqslant D$.

Proof idea: Pure states are extremal and $\|\psi\psi^*\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_M}} = \|\psi\|^2_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_M}}$.

Outline

[Notation from now on: $\|\cdot\| := \|\cdot\|_{\ell_2^n}$, $\|\cdot\|_\varepsilon := \|\cdot\|_{(\ell_2^d)^{\otimes_\varepsilon M}}$, $\|\cdot\|_\pi := \|\cdot\|_{(\ell_2^d)^{\otimes_\pi M}}$.]

Fact: For any unit vector $\psi \in (\mathbf{C}^d)^{\otimes M}$, $\|\psi\|_\varepsilon \geqslant \frac{1}{\sqrt{d^{M-1}}}$, i.e. $E(\psi) \leqslant (M-1)\log d$.

Question: Are multipartite pure states generically 'very' or 'little' entangled?
$\longrightarrow$ For a unit vector $\psi \in (\mathbf{C}^d)^{\otimes M}$ sampled at random, what is typically the value of $\|\psi\|_\varepsilon$, and thus of $E(\psi)$?

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Theorem [Typical injective norm of a uniformly distributed unit vector (Aubrun/Szarek)]

There exist constants $c, C, c_0 > 0$ s.t., for $\psi \in (\mathbf{C}^d)^{\otimes M}$ a uniformly distributed unit vector,
$\hookrightarrow$ i.e. $U\psi \sim \psi$ for all unitary U on $(\mathbf{C}^d)^{\otimes M}$

$$\mathbf{P}\left(c\sqrt{\frac{M\log M}{d^{M-1}}} \leqslant \|\psi\|_\varepsilon \leqslant C\sqrt{\frac{M\log M}{d^{M-1}}} \right) \geqslant 1 - e^{-c_0 dM\log M}.$$

Consequence: For $\psi \in (\mathbf{C}^d)^{\otimes M}$ a uniformly distributed pure state, when d is large, $E(\psi) = (M-1)\log d - \log(M\log M) + O(1)$ with high probability.
$\longrightarrow$ Such random multipartite pure states are typically close to maximally entangled.

In fact, for $M > 2$, there are no explicit examples of so highly entangled multipartite pure states!

Main tools in the proof

Observation: A uniformly distributed unit vector $\psi \in (\mathbf{C}^d)^{\otimes M}$ has the same distribution as $g/\|g\|$, where $g \in (\mathbf{C}^d)^{\otimes M}$ is a Gaussian vector.
$\hookrightarrow$ independent complex Gaussian entries with mean 0 and variance 1

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• **Estimating the supremum of a Gaussian process:**

Given $K \subset \mathbf{C}^n$ a subset of the unit sphere, we want to estimate $\mathbf{E} \sup_{x \in K} |\langle x | g \rangle|$.

Definitions: $C_\delta \subset K$ is a δ-*covering set* if: $\forall\, x \in K,\ \exists\, y \in C_\delta : \|x - y\| \leqslant \delta$.
$S_\delta \subset K$ is a δ-*separated set* if: $\forall\, x, y \in S_\delta,\ x \neq y \ \Rightarrow\ \|x - y\| \geqslant \delta$.

Fact: Upper bound: $\mathbf{E} \sup_{x \in K} |\langle x | g \rangle| \leqslant \frac{1}{1-\delta} \mathbf{E} \sup_{x \in C_\delta} |\langle x | g \rangle| \leqslant \frac{1}{1-\delta} \sqrt{2 \log |C_\delta|}$

discretization argument

'baby' chaining argument

Lower bound: $\mathbf{E} \sup_{x \in K} |\langle x | g \rangle| \geqslant \mathbf{E} \sup_{x \in S_\delta} |\langle x | g \rangle| \geqslant \delta \sqrt{\frac{\log |S_\delta|}{2\pi \log 2}}$

Sudakov inequality

$\longrightarrow$ Taking $K := \left\{ \varphi_1 \otimes \cdots \otimes \varphi_M \,:\, \varphi_i \in \mathbf{C}^d,\ \|\varphi_i\| = 1 \right\}$, we have $\mathbf{E}\|g\|_\varepsilon = \mathbf{E} \sup_{\varphi \in K} |\langle \varphi | g \rangle|$.

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• **Gaussian concentration inequality:**

Given $f : \mathbf{C}^n \to \mathbf{R}$ an L-Lipschitz function, we have: $\forall\, \varepsilon > 0,\ \mathbf{P}\left(f(g) \gtrless \mathbf{E} f \pm \varepsilon\right) \leqslant e^{-\varepsilon^2/L^2}$.

$\longrightarrow$ Taking $f := \|\cdot\|_\varepsilon$, f is 1-Lipschitz.

Conclusion and follow-up questions

Result: For all $M \geqslant 2$, there exist close to maximally entangled M-partite pure states.
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In order to obtain such a state, picking it uniformly at random works with high probability!

Questions:

1. Can we exhibit so highly entangled states with constructions that require less randomness? or even explicit ones?
2. The uniform distribution might not capture truly 'interesting' states... What about estimating the typical entanglement of more 'physically relevant' states?

Outline

The curse of dimensionality when dealing with many-body quantum systems

Main issue when dealing with many-body quantum systems: exponential growth of the dimension. However, 'physically relevant' states of such systems are often well approximated by so-called *tensor network states (TNS)*, which form a small subset of the global state space.

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Example: A *matrix product state (MPS)* on $(\mathbf{C}^d)^{\otimes M}$ is a pure state $\psi \in (\mathbf{C}^d)^{\otimes M}$ of the form

$$\psi = \sum_{k_1,\ldots,k_M=1}^{d} \mathrm{Tr}\left(X_{k_1}^{(1)} \cdots X_{k_M}^{(M)} \right) e_{k_1} \otimes \cdots \otimes e_{k_M}, \text{ where } X_1^{(i)},\ldots,X_d^{(i)} \in \mathcal{M}_q(\mathbf{C}),\ 1 \leqslant i \leqslant M.$$

$\longrightarrow$ Such state is described by Mdq^2 parameters, which is linear rather than exponential in M.

[Vocabulary: d is the *physical dimension*. q is the *bond dimension*.]

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Fact: On a 1D system (M subsystems disposed on a line), the *ground state of a gapped local Hamiltonian* is well approximated by an MPS (Hastings, Landau/Vazirani/Vidick)

spectral gap lower bounded by a constant independent of M

composed of terms which act non-trivially only on nearby sites

$\longrightarrow$ In condensed-matter physics, MPS are used as Ansatz in ground energy computations: optimization over a manageable number of parameters, even for large M.

independent complex Gaussian entries with mean 0 and variance 1

Idea: Pick $X_1^{(i)},\ldots,X_d^{(i)} \in \mathcal{M}_q(\mathbf{C})$, $1 \leqslant i \leqslant M$, independent Gaussian matrices.
Let $\psi \in (\mathbf{C}^d)^{\otimes M}$ be the corresponding random MPS, i.e.

$$\psi = \frac{\psi'}{\|\psi'\|} \text{ with } \psi' = \sum_{k_1,\ldots,k_M=1}^{d} \mathrm{Tr}\left(X_{k_1}^{(1)}\cdots X_{k_M}^{(M)}\right) e_{k_1}\otimes\cdots\otimes e_{k_M}.$$

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Remark: The parameter q quantifies the amount of bipartite entanglement: across any splitting of subsystems $\{1,\dots,L\}$ vs $\{L+1,\dots,M\}$, ψ has Schmidt rank at most $q^2 \ll d^L$.

area vs volume law

Now what about genuinely multipartite entanglement?
$\longrightarrow$ If $q=1$, $\psi=\psi_1\otimes\cdots\otimes\psi_M$ is separable. But what can we say for $q\gg 1$?

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Result: For $\psi \in (\mathbf{C}^d)^{\otimes M}$ a Gaussian MPS with bond dimension q, when d and q are large, $E(\psi) = (M-1)\log\min(d,q) + O(1)$ with high probability (Fitter/Lancien/Nechita).
$\longrightarrow$ Such random multipartite pure states have an amount of entanglement which is not maximal if $q \ll d$, but still extensive.

Proof idea: Estimate $\mathbf{E}\|\psi\|_\varepsilon$ with a discretization argument and bound the probability of deviating from it with a local version of the Gaussian concentration inequality.

- Can we estimate the injective norm for *other models of random multipartite pure states*?
  $\longrightarrow$ Trade-off between 'mathematically tractable' and 'physically relevant'...!
  on regular lattice or more complicated graph
  E.g. 2D or 3D ground states are also well-approximated by tensor network states.
  $\longrightarrow$ Some of their typical entanglement-related properties have been studied: *correlations* (Lancien/Pérez-García), *mutual information* (Hayden/Nezami/Qi/Thomas/Walter/Yang), etc.

Perspectives

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- What about computing *projective norms* rather than injective norms?
  This would be particularly useful for quantifying the entanglement of mixed multipartite states, which is in general a *hard problem* (Gharibian, Pérez-García).

  For pure states, we at least have a lower bound (by duality): $\|\psi\|_\pi \geqslant \frac{1}{\|\psi\|_\varepsilon}$.

  For mixed states, we can define *sufficient conditions for entanglement*, which consist in checking pure state entanglement (Jivulescu/Lancien/Nechita).

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